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Arbitrary Rotated Coordinate Systems for the Inclined Plane as an Introduction to Group Theory in the Introductory Physics Classroom

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ARBITRARY ROTATED COORDINATE SYSTEMS FOR THE INCLINED PLANE AS AN INTRODUCTION TO GROUP THEORY IN THE INTRODUCTORY PHYSICS CLASSROOM

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ABSTRACT

The elementary problem of a block sliding down an inclined plane is examined in detail with respect to different oriented coordinate systems that are typically not used due to the complexity of the problem. After solving for the equation of motion in these different coordinate systems group theory is applied and shown to yield the same results.

INTRODUCTION

This problem is directly influenced by Galileo Galilei (Galileo, 1941) who in the 17th century designed and carried out experiments to confirm that the earth's acceleration is constant by using the inclined plane. Accurate timing methods were an issue in the 17th century and, if Galileo could slow down the acceleration of an object, it would be easier to measure. Galileo observed that the acceleration of an object was constant at different inclination angles. He concluded that in the limit where the inclination angle is at ninety degrees, the acceleration due to gravity was indeed a constant.

It is the experience of one of the authors that students have several issues associated with this canonical problem. The issue that this paper focuses on is that of the standard technique of rotating the coordinate system with respect to the inclined plane. Since there is no absolute coordinate system, one is free to choose any of an infinite number of coordinate systems. In this paper we will confine ourselves to two-dimensional Cartesian coordinate systems that are rotated with respect to one another but all share a common origin. Of course these are all isomorphic situations when considering rotational transformations, but most introductory physics students will not have the linear algebra skills to use a two-dimensional rotation matrix. To understand the application of rotational transformations, one must note that the magnitude of the acceleration vector will remain the same for any given rotated coordinate system. The components of the acceleration vector will adjust accordingly to the respective coordinate system used to solve the system.

This is the familiar situation one finds in the traditional texts (Halliday et al. 2013). The coordinate system is rotated (or as some texts say *tilted*) such that the x-axis is parallel with the surface of the inclined plane and the y-axis is perpendicular to the surface of the inclined plane (see Figure 1). This judicious choice (or tricky insight as some students may retort) of coordinate systems allows the acceleration and the normal vectors to be parallel to the x- and y-axes respectively. This means that only the force due to gravity will have to be broken into its x- and y-components. Once this is done, it is easily seen that the motion occurs in one dimension.

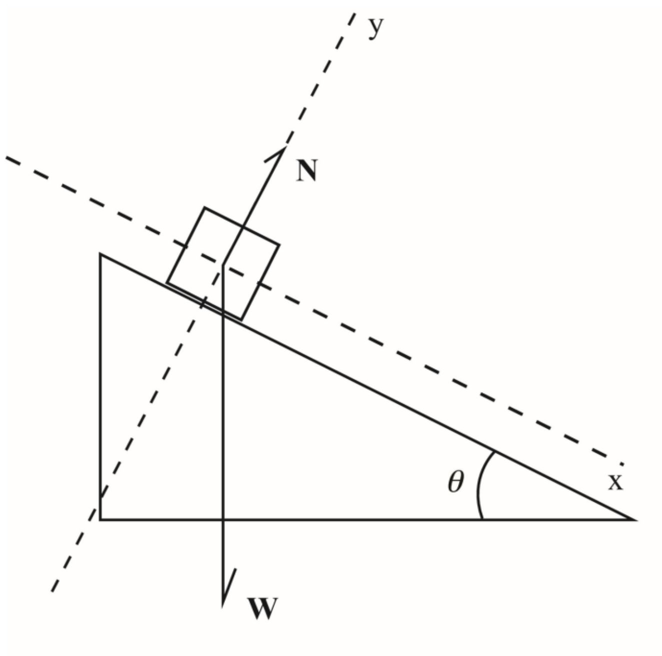

Figure 1. Above is the typically rotated coordinate system and free body diagram used in solving for the acceleration of a block (modeled as a point particle) sliding down an inclined plane with inclination angle θ.

For completeness, Newton's Second Law (for an object of constant mass) in this situation is found to be

$$\sum F_x = mgsin\theta = ma$$
$$\sum F_y = N - mgcos\theta = 0$$

where F_x and F_y are the x and y components of the forces, m is the mass of the point particle, g is the magnitude of the earth's gravity, N is the normal force, and θ is the angle of the inclined plane. This choice of coordinate system allows the x-component of the acceleration to be the only component one needs (hence dropping any subscript of direction for the acceleration). The block is in equilibrium in the y-direction and thus this is a static component resulting in zero on the right hand side of Newton's Second Law. This leads to the famous result

$$a = gsin\theta.$$

This unrotated coordinate system (see Figure 2) is the one many students will naturally choose left to their own devices without input from a text or instructor. Many times it seems as though students are frustrated trying to solve the problem in this more complicated situation. Upon being shown that a rotated coordinate system makes the problem much easier to solve, the students reason that the result might be dependent on the orientation of the coordinate system which is simply not the case.

Once again, writing down Newton's Second Law, but this time the acceleration and normal force will have to be broken up into individual components, leads to

$$\sum F_x = N_x = ma_x$$
$$\sum F_y = N_y - W = ma_y$$

where a_x and a_y are the x- and y-components of the acceleration and N_x and N_y are the x- and y-components of the normal force. The components can be written in trigonometric terms,

$$\sum F_x = N sin\theta = ma cos\theta \tag{1}$$

$$\sum F_y = N cos\theta - mg = -ma sin\theta. \tag{2}$$

Solving Eqn. 1 for the normal force in order to remove it from the system of equations

$$N = ma\frac{cos\theta}{sin\theta}, \tag{3}$$

taking Eqn. 3 and substituting into Eqn. 2 to obtain an equation all in terms of mass, acceleration, gravity, and the inclination angle

$$ma\frac{cos\theta}{sin\theta}cos\theta - mg = -ma sin\theta,$$

multiplying through by $sin\theta$ and rearranging to get the mass and acceleration on one side

$$mg sin\theta = ma sin^2\theta + ma cos^2\theta,$$

and using the identity that $sin^2\varphi + cos^2\varphi = 1$ leads to

$$mg sin\theta = ma$$

where one finds the familiar result,

$$a = g sin\theta.$$

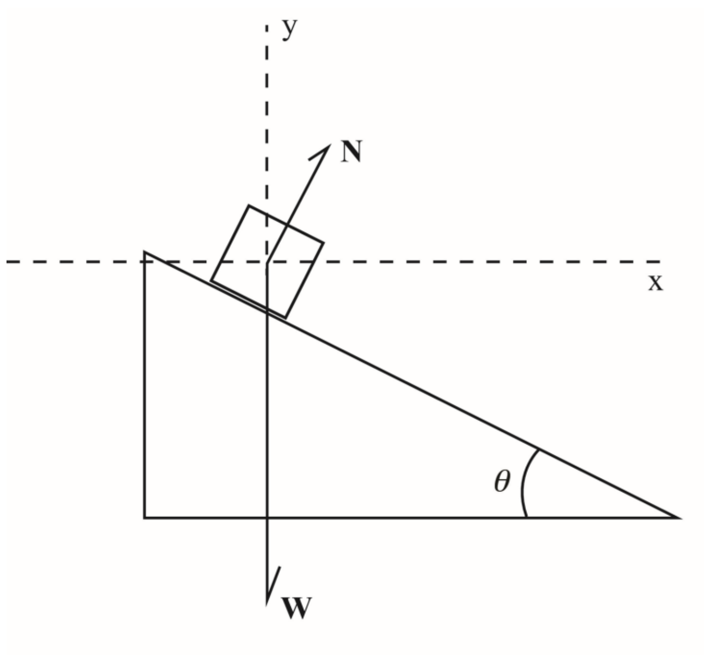

Figure 2. A block (modeled as a point particle) sliding down an inclined plane of inclination angle θ with the free body diagram. An unrotated coordinate system that many students initially choose for this problem is shown.

In this section, an arbitrarily rotated coordinate system is considered (see Figure 3). The coordinate system is rotated at a positive angle α with respect to the weight vector.

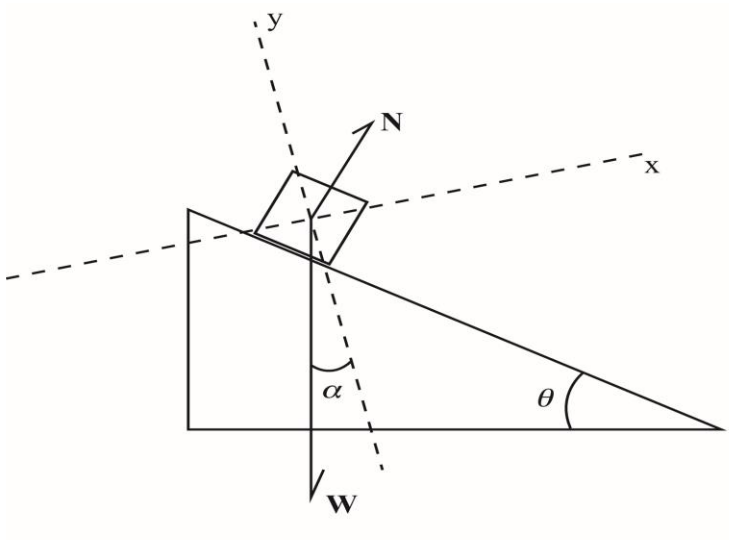

Figure 3. A block of mass m sliding down an inclined plane of inclination angle θ with the free body diagram. An arbitrarily counter clockwise rotated coordinate system of angle α with respect to the weight vector is shown.

Rewriting the force and acceleration in terms of x- and y-components leads to

$$\sum F_x = N_x - W_x = ma_x$$
$$\sum F_y = N_y - W_y = ma_y.$$

The components can all be written in trigonometric terms of the angle of the incline and the coordinate system. This leads to

$$\sum F_x = Nsin(\theta + \alpha) - mgsin\alpha = macos(\theta + \alpha) \tag{4}$$
$$\sum F_y = Ncos(\theta + \alpha) - mgcos\alpha = -masin(\theta + \alpha). \tag{5}$$

Solving Eqn. 4 for the normal force leads to

$$N = mg\frac{sin\alpha}{\sin(\theta+\alpha)} + ma\frac{\cos(\theta+\alpha)}{\sin(\theta+\alpha)}. \tag{6}$$

Plugging Eqn. 6 into Eqn. 5, one finds,

$$\left(mg\frac{sin\alpha}{\sin(\theta+\alpha)} + ma\frac{\cos(\theta+\alpha)}{\sin(\theta+\alpha)}\right)\cos(\theta + \alpha) - mgcos\alpha = -masin(\theta + \alpha). \tag{7}$$

Multiplying through to get rid of the denominator in Eqn. 7 leads to

$$mgsin\alpha\cos(\theta + \alpha) = macos^2(\theta + \alpha) - -mgcos\alpha\sin(\theta + \alpha) = -masin^2(\theta + \alpha).$$

Once again using $sin^2\vartheta + cos^2\vartheta = 1$, one determines that

$$mgsin\alpha\cos(\theta + \alpha) - mgcos\alpha\sin(\theta + \alpha) = -ma.$$

Using the angle addition formulas for sine and cosine on the right hand side of the equation leads to

$$mgsin\alpha cos\theta cos\alpha - mgsin\alpha sin\theta sin\alpha - mgcos\alpha sin\theta cos\alpha$$
$$+mgcos\alpha cos\theta sin\alpha = -ma.$$

Simplifying and combining terms leads to

$$-mgsin\theta(sin^2\alpha + cos^2\alpha) = -ma.$$

Once again using $sin^2\vartheta + cos^2\vartheta = 1$, one finds the celebrated result,

$$a = gsin\theta.$$

Examining the situation of a point mass m sliding down an inclined plane of inclination angle θ, it has been explicitly demonstrated in the rotated, unrotated, and arbitrarily rotated coordinate systems, that the magnitude of the acceleration is the same. It is the hope of these authors that this work will bring a deeper insight to future introductory physics students with respect to Newton's Second Law and rotated coordinate systems.

FURTHER DISCUSSION

In the preceding section, the rotation matrix was actually employed even though it was not explicitly stated. This is an opportunity for instructors in introductory physics courses to introduce this ubiquitous device. This matrix

$$\begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix} \tag{8}$$

rotates the coordinate system about the origin through a counter clockwise angle α with respect to the original positive x-axis. It is the fact that this rotation matrix preserves the lengths that allows one to find the accelerations in the previous sections to all be the same.

Besides introducing this rotation matrix, instructors can introduce the idea of a group to students. A group (Gallian 1974), denoted as G, is a set of elements $\{e,f,g,h,k,\ldots\}$ together with a binary operator denoted by $\circ$. The result of the binary operations is subject to the following four requirements:

- Closure: if f,g are elements of the group G, then $f\circ g$ is an element of the group G,
- Identity elements: there exists an identity element e in the group G such that $e\circ f = f\circ e = f$ for any element f that is a member of the group G,
- Inverses: for every f that is a member element of G there exists an inverse element f^{-1} that is a member element of G such that $f\circ f^{-1} = f^{-1}\circ f = e$ and where e is the identity element described in preceding requirement,
- And Associative Law: the law that $f\circ(g\circ h) = (f\circ g)\circ h$.

Examples of groups are the integers under addition, the real numbers under multiplication, and rotations and reflections of an equilateral triangle that leaves the triangle invariant.

The rotation matrix is one representation of the SO(2) group. The SO(2) group is the group of 2 × 2 matrices with determinant equal to one and whose inverse is the matrix's transpose. The determinant being equal to one ensures that only proper rotations can occur and thus chirality (or *handedness* is preserved). One representation of SO(2) is that as shown in Eqn. 8,

$$\mathbf{R} = \begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix}.$$

Note that $\mathbf{RR}^{-1} = \mathbf{RR}^{\mathbf{T}} = \begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix}\begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as required by the orthogonality requirement and the determinant of **R** = $\begin{vmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{vmatrix}$ = cos²α + sin²α = 1 as required by the special requirement.

The matrix operation between **v** = $\begin{pmatrix} x \\ y \end{pmatrix}$ and the rotation matrix **R** through some angle α is

$$\mathbf{Rv}=\begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xcos\alpha - ysin\alpha \\ xsin\alpha + ycos\alpha \end{pmatrix}.$$

The application of the rotation matrix to our incline problem is to look at Figure 2 where the coordinate system is nonrotated. In this coordinate system the weight **W** is aligned with the negative y-axis thus one can write the weight vector as **W** = $\begin{pmatrix} 0 \\ -mg \end{pmatrix}$ and the normal vector **N** is broken up along the x- and y-axes such as **N** = $\begin{pmatrix} N_x \\ N_y \end{pmatrix}$. Now apply a rotation of the coordinate system through an angle θ as shown in Figure 1. This leads to a new weight vector **W′** and new normal vector **N′** as shown explicitly below,

$$\mathbf{W'} = \mathbf{RW} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ -mg \end{pmatrix} = \begin{pmatrix} mgsin\theta \\ -mgcos\theta \end{pmatrix}$$

and

$$\mathbf{N'} = \mathbf{RN} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} N_x \\ N_y \end{pmatrix} = \begin{pmatrix} N_x cos\theta - N_y sin\theta \\ N_x sin\theta + N_y cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} N(-sin\theta)cos\theta - N(-cos\theta)sin\theta \\ N(-sin\theta)sin\theta + N(cos\theta)cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ N \end{pmatrix}.$$

These are the same results as shown in the first section of this paper.

This elementary example of a block modeled as a point particle sliding down an inclined plane can be used to demonstrate a host of physical phenomena. The importance of the mass not appearing in the equation of motion is often discussed. It is the authors hope that this example may also be used to show that the physics results are independent of the coordinate systems as well as an early example of group theory in the survey of physics course.

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