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CONNECTING ALGEBRA AND GEOMETRY TO FIND SQUARE AND HIGHER ORDER ROOTS

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ABSTRACT

Unfortunately, the repeat rate for all core curriculum courses including calculus-I and II, and also math education courses is too high. K-12 students are barely able to do estimation, an important part of Mathematics, whether it is adding fractions or finding roots, or for that matter, simple percent problems. In this paper we present geometric ways to reason about approximation of square roots and cube roots that are not accessible with simple routine techniques. We combine graphical methods, the use of a geometry software (sketch pad) and convergence of sequences to find higher order roots of positive real numbers and in the process, reason with recursion.

Keywords: square roots, higher order roots, geometry sketchpad, sequences, convergence, iterations, recursion.

INTRODUCTION

This paper is about recursive reasoning in the context of root extraction written with teachers in mind who do not allow calculators in their classes. But, as we all know, these days most students grab their calculators in a hurry. Also, those who have no calculator, at least a couple of them, would tell you they could not find the sum as they forgot their calculator at home. We have personally witnessed this scenario from college algebra to higher level math classes and throughout our math education classes. Unfortunately, a fix of this problem is not on the horizon any time soon.

An internet search for a quick way to find roots resulted in a few methods, mostly based on *Vedic Mathematics*. Vedic Mathematics is old Indian mathematics rediscovered from Vedas (the oldest Hindu religious scriptures) between 1911 and 1918 by Bharti Ksna Thirthji (1884 – 1960). This method generally requires students to know the squares or cubes of the numbers 1 – 9, and possibly 11 – 19, depending on how large of a number you are considering to find a root for. If the given number is a perfect square or a cube of a number, then the examples presented on the internet offer good solutions using Vedic Mathematics.

In this paper we present neat geometric ways to reason about estimation of square roots and cube roots that are not accessible with simple routine techniques. We will show that solutions can be found by combining graphical methods, simple recursive reasoning and the use of a geometry software (sketch pad).

Students in high school or in college algebra should be able to understand the methods presented and use them for their own problem solving activities.

In our first example we present a simple recursive sequence graphically, with the aim to argue what the ultimate value (the limit) should be. In the second example, we present a method to find roots of *any order*, again using recursion and averages. The examples have the same basic solution structure that a mathematician could recognize as an application of the Banach Contraction Principle (stated at the end of the paper), but without all the advanced mathematical reasoning.

MATHEMATICAL REASONING

In the summer and fall of 2013 we interviewed more than 30 students including students registered as math majors and students in the math education courses at a university in the southern USA. The objective was to assess the mathematical reasoning they used to solve problems at the college algebra level. We noticed that many of these students, including the mathematics majors, used various preconceived ideas and patterns they got ingrained in their heads since their K-12 classes, to get their solutions. They used illogical reasoning in a haphazard way to find a solution, whether right or wrong. Graphs were not a regular feature of their arguments and deduction was even rarer. It was clear to us that students at all levels needed more instruction in how to construct reasoning, how to use mathematical tools and how to deepen their knowledge for problem solving. Of course as every mathematician knows, it is a huge task to remedy this situation and a single article will not be enough. However, we believe we should try and continue in the proper direction.

We asked the students a simple recursive algorithm: given a sequence $\{x_n\}$ defined as $x_{n+1} = \frac{1}{2}x_n + 2$, find successive values of x_n using $x_1 = 6$.

To test the students we asked the following questions:

- Does the sequence converge, that is: Do the values get closer and closer to a fixed number as the values of n get larger?
- If it does, can you tell to what number?

Several students were at a complete loss. The students who were willing to try were asked to make a conjecture for the ultimate value of x_n as n becomes large. Most of the students who got the correct answer had little understanding of what really was happening, why the values were getting closer and closer to each other, and finally reaching to a limit of 4.

GEOMETRIC METHOD

We present a geometric (graphical) method, as shown below in Figure 1, for students to develop a better understanding of the limit of the recursive sequence. Since on the line $y = x$, the x and y values are always equal, we use the graph of this line to represent recursion. First we graph the functions $F(x) = \frac{1}{2}x + 2$ and $H(x) = x$. The function $H(x) = x$ is used to visualize recursion and to support the reasoning. Next we generate a sequence x_n recursively (graphically) as shown in Figure 1, with $x_1 = 6$ as our starting point. The graphs serve as a visualization of how to find a new value of x_n from its previous value. The key component in

recursion is the graph of $H(x) = x$. The red path in the graph from points A to B, and B to C to D...E... shows how the recursion is taking place starting with $x_1 = 6$.

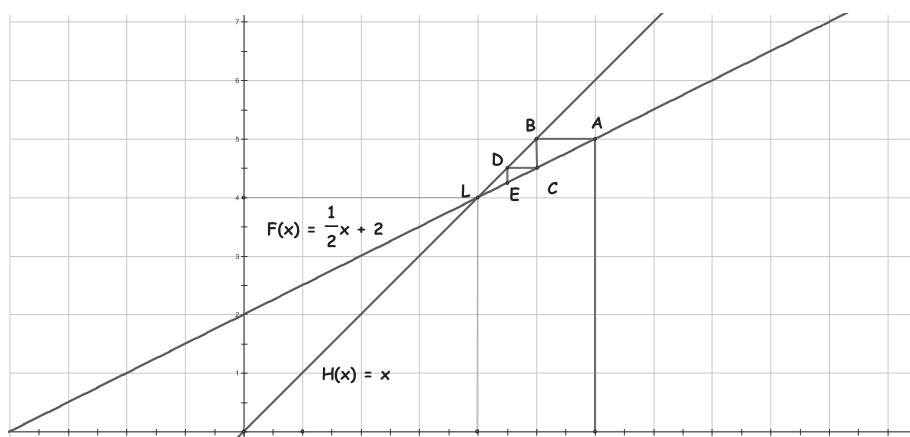


Figure 1. The recursive values of the sequence in the graph $F(x) = \frac{1}{2}x + 2$ using $H(x) = x$.

The recursive process can easily facilitate students in making a conjecture about the final value of x in just a few iterations. It is fascinating to see how the values of $H(x) = x$ are transferred to $F(x)$ and vice versa to continue the recursive process. Moreover, this graphical process may help a student to understand the algebraic process better, in the sense that the iterative process does converge to the point of intersection of the graphs of $F(x)$ and $H(x)$.

FINDING SQUARE AND HIGHER ORDER ROOTS

In 2005 Bryan Dörner (1) published a paper in the College Mathematics Journal on methods for finding square, cube and higher degree roots with techniques that have origins he traced back to Hellenistic geometry and ancient Indian algebra. In his paper he described a method for finding any degree roots with matrices and vectors. He also mentioned another mathematician, Wassell (2), who used arithmetic means (the averages) instead of the geometric mean to find square roots (as shown below). For example, assume we want to find square root of a number N (not necessarily an integer). Wassell (2) starts with an approximate value a , and then computes $b = \frac{N}{a}$. His next guess is the arithmetic average of a and b , which is $c = \frac{a+b}{2}$. He then repeats this process, first finding $d = \frac{N}{c}$, and then calculating the average of c and d : $e = \frac{c+d}{2}$ as his next guess. He then continues the recursive calculations. This process converges fast in just a few iterations and your initial guess could be *any number* smaller than N . Since students nowadays dislike fractions they may quickly give up on this algebraic process. However, we believe it is to their advantage to carry out these

iterations algebraically in order to gain a better understanding of the procedure as suggested by Wassell. They will learn how fast this process converges. Dorner's paper describes Wassell's idea of averages for finding square roots. We thought it would work for finding higher degree roots as well, an application we could not find in Dorner's paper (1).

Averages and Recursion for Higher Degree Roots

Assume we want to find *cube root* of a real number $N = 285.33$ (the calculator gives 6.583383361 as an answer). We begin with an initial estimate, say $X_1 = 5$, which of course is a raw estimate of cube root of N . However, the algorithm is amazing. In just a few iterations it provides an estimate very close to the actual answer. As we did earlier, we first calculate $\frac{285.33}{5} = 57.066$. We continue the division still using the same initial estimate of 5: $\frac{57.066}{5} = 11.4132$. Now we find the average of the *three* numbers 5, 5, and the newly calculated number 11.4132: $X_2 = \frac{1}{3}(5 + 5 + 11.4132) = 7.13773333$. This is our new estimate of the cube root of the number $N = 285.33$. With this new value the process is repeated every time using the average of three numbers: Find $X_3 = \frac{1}{3}(7.13773333 \dots + 7.13773333 \dots + \frac{285.33}{7.13773333^2}) = 6.625322211$ which is not too far from the actual answer. We will show that in just a couple of iterations we will get an answer that is identical to the actual answer up to seven decimal places. Tabulating these values we obtain:

$$X_3 = 6.625322212 \text{ ————— } \frac{285.33}{6.625322212^2} = 6.500300407$$

$$X_4 = \frac{1}{3}(X_3 + X_3 + \frac{285.33}{6.625322212^2}) = 6.583648277$$

$$X_4 = 6.583648277 \text{ ————— } \frac{285.33}{6.583648277^2} = 6.58285356$$

$$X_5 = \frac{1}{3}(X_4 + X_4 + \frac{285.33}{6.58285356^2}) = 6.583383371$$

Actual cube root to nine decimal places is: $\sqrt[3]{285.33} \approx 6.583383361$

Geometry Sketchpad

We will now try to find a geometric connection to the above algebraic formula using a geometry sketch pad convergence of a sequence. We will use the same method presented in the first part of this paper: Let N be any real number. We define a sequence of numbers recursively using X_{n+1} so that $X_{n+1} = \frac{1}{3}(2X_n + \frac{N}{X_n^2})$. We will now show graphically the recursive process. First graph the functions $F(x) = \frac{1}{3}(2x + \frac{N}{x^2})$ and $H(x) = x$. Let $N = 285.33$. We begin with an initial guess of $x = \frac{1}{3}15$, and follow it on the graph as in the first part of this article with the path from point A to point B, then to points C, D.... From the

graph in figure 2 it is clear that this path converges to the point of intersection of the graphs of the functions $F(x)$ and $H(x)$, rather quickly. The same recursive method works for odd roots, but since even roots of negative numbers are complex numbers we will not discuss this presently.

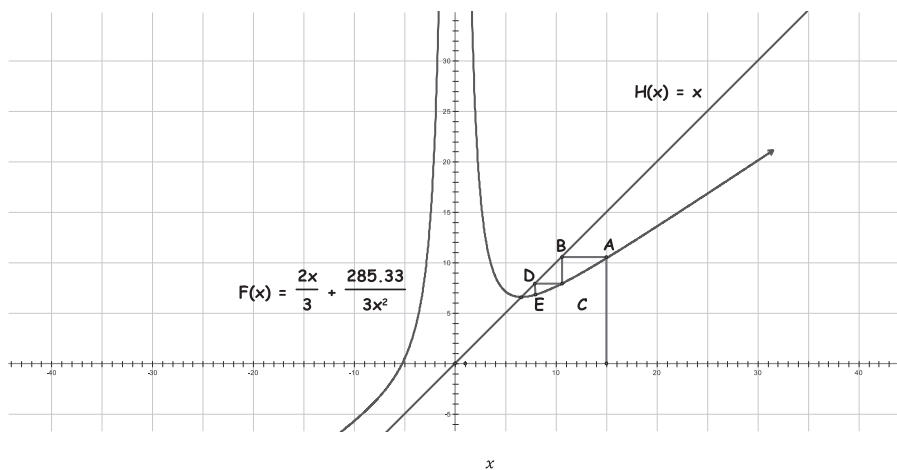


Figure 2. Recursive process shown graphically: $F(x) = \frac{2x}{3} + \frac{285.33}{3x^2}$ and $H(x) = x$; $x > 0$.

Algebraically, solving equations $F(x) = \frac{1}{3}(2x + \frac{285.33}{x^2})$ and $H(x) = x$ simultaneously while equating them to find the point of intersection of these two functions, we obtain $2x + \frac{285.33}{x^2} = 3x$. It follows that $x^3 = 285.33$. Therefore $x = \sqrt[3]{285.33} \approx 6.583383361$ in nine decimals.

We have shown that with some graphical support we can find cube roots of any number with ease. The arguments do not constitute a formal proof, but this was not our intention. These results are the products of non-routine reasoning and with tools that support visual thinking and are rooted in deductions. Continuing, we propose the recursive formula to find the 4th root of a real number. Let N be any real number. We define a sequence of numbers recursively using

$$X_{n+1} = \frac{1}{4}(3x_n + \frac{N}{x_n^3}).$$

If we solve the equation $x = \frac{1}{4}(3x + \frac{N}{x^3})$, we obtain $4X^4 = 3X^4 + N$, which in turn gives $x^4 = N$, or $x = \sqrt[4]{N}$.

We can obtain a geometric solution using iterations as above, graphing two functions, $F(x) = \frac{1}{4}(3x + \frac{N}{x^3})$ and $H(x) = x$ on the same axes. We then choose an estimate for fourth root of N to be any real number smaller than N , and continue the recursive process as above, after making an initial guess for the fourth root of N . Generalizing: To find the $(n + 1)$ th root of a real number N , we graph the equations $F(x) = \frac{1}{(n+1)}(nx + \frac{N}{x^n})$ and $H(x) = x$ and use an iterative process to find the $(n + 1)$ th root of N , first *geometrically* and then *algebraically*.

SUMMARY

We have established a simple iterative process to show a connection between algebra and geometry to find the n^{th} root of any positive real number. As the even roots of a negative number give complex numbers, we have concentrated only on positive real numbers. It is nice to see how geometry could play a vital role in showing students the actual convergence of a sequence of numbers to cube roots, which students only perceive as an algebra problem. The NCTM (3) has advocated for the right reasons to show students a connection between algebra and geometry whenever possible. The authors tried to show this connection and have to admit that it was also a learning process for them.

The interested explorer can construct her/his own activity with higher degree roots, and answer questions such as: What are the limitations for this process? What does it teach the young students? We believe a teacher can find more about the deeper connections with more advanced mathematics, in this case Banach's Contraction Principle: If X is a closed subset of \mathbb{R}^n and $F: X \rightarrow X$ satisfies the condition that $\|F(x) - F(y)\| \leq K \|x - y\|$ with $K < 1$, then there is a unique point p of X , such that $F(p) = p$; moreover for any point x of X the sequence $F(x), F(F(x)), F(F(F(x))), \dots$ converges to the point p , ($\|x\|$ defines the norm of X , which in our case is the absolute value of x). Of course there are many avenues for explorations. We have just presented one.

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