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ABSTRACT

A standard topic in a College Algebra course is to find the coordinates of the midpoint of a line segment connecting two points. In the early twentieth century [1] and [2], in a few texts, this idea was extended to find the coordinates of a point that divides a line segment in any given ratio. In this paper, we present a neat application of this extended idea, and further use it to find the co-ordinates of the centroid of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Key Points: Mid Point, Weighted Average, the Centroid

INTRODUCTION

A standard topic in a College Algebra course is to find the coordinates of the midpoint of a line segment connecting two points (x_1, y_1) and (x_2, y_2). The midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, which is the *average* of the x's and the y's. A natural extension of this idea is to find the coordinates of a point that divides the segment connecting the points (x_1, y_1) and (x_2, y_2) in the ratio a : b, where 'a' and 'b', are any two real numbers. The coordinates of such a point are in fact $\left(\frac{ax_2+bx_1}{a+b}, \frac{ay_2+by_1}{a+b}\right)$, the *weighted average* of x's and the y's, with weights 'a' and 'b', as expected.

A constant struggle to teach students the midpoint formula and not the general formula (the one using the ratio a : b), is that students invariably confuse the midpoint formula with either the slope formula or with the distance formula, or with both. If a more general formula is taught, then hopefully, it would mitigate the confusion of students in using the three formulas; the midpoint formula, the formula for slope of a line, and the distance formula. We also note that the formula for the midpoint becomes a corollary of the general formula with both a = 1, and b = 1.

In this paper, we will first present an example using the generalized formula, then for the same example, we will show the use of this formula to find the midpoint. We will present a neat application of this formula that would usually be solved using an equation of a straight line.

Finally, in the Appendix A, we will show that the co-ordinates of the centroid of a triangle connecting three $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ points are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$. We will include the proof of the general formula.

The proof for this formula shows a nice connection between algebra and geometry. The Standards of NCTM have consistently emphasized to teach mathematics as a whole instead of teaching it as a collection of segregated areas; *i.e.*, it has advocated to show *connection(s)* between different branches of mathematics.

Example 1. Find the coordinates of the point that divides the segment connecting the points (-2, 4) and (6, 10) in the ratio of 3 : 4. Then find the midpoint using the same formula and compare your result using the standard midpoint formula.

Solution: Using the formula $\left(\frac{ax_2+bx_1}{a+b},\frac{ay_2+by_1}{a+b}\right)$, with a = 3, and b = 4, we obtain

$$x=\frac{3(6)+4(-2)}{3+4}=\frac{18-8}{7}=\frac{10}{7},$$

$$\text{and } y=\frac{3(10)+4(4)}{3+4}=\frac{30+16}{7}=\frac{46}{7}$$

Thus the point dividing the segment joining the points (-2, 4) and (6, 10) in the ratio 3 : 4 is $\left(\frac{10}{7},\frac{46}{7}\right)$.

In order to find the midpoint, we substitute a = 1, and b = 1, in the above formula to obtain:

$$x=\frac{1(6)+1(-2)}{1+1}=\frac{6-2}{2}=\frac{4}{2}=2,$$

$$\text{and } y=\frac{1(10)+1(4)}{1+1}=\frac{10+4}{2}=\frac{14}{2}=7$$

Thus the midpoint is (2, 7).

Finally using the traditional midpoint formula, the midpoint is,

$$x=\frac{6-2}{2}=\frac{4}{2}=2,\quad y=\frac{10+4}{2}=\frac{14}{2}=7.$$

Hence the coordinates of the midpoint computed by the generalized formula or by the traditional formula are the same. This suggests that the generalized formula might be correct. However, we know that in mathematics an example is not sufficient to accept that the result holds in its generality. Thus we include a proof of the generalized formula in the Appendix A.

We next present an application using the generalized formula above.

Application: Statically it is known that in 1954, the record of running a mile was 4 minutes and by 1975, the record time has fallen to 3 minutes and 50 seconds. Assuming that the record had fallen at a constant rate, find the record time in the year 1960.

Solution: This type of problem is usually done by first finding the equation of a line through two points and then substituting the x-value, to find the y-value. Let us verify if using our formula above can give us the same result. For simplification, let us label 1954 as zero and convert minutes into seconds. Thus our two points are (0, 240) and (21, 230). The year 1960 is 6 units from 1954, and 15 units from 1975. Thus the ratio is 6 : 15 *or* 2 : 5. Since the x-value is already known, we only need to find the y-value. Using the above formula, we obtain,

$y = \dfrac{2(230)+5(240)}{2+5} = \dfrac{1660}{7} \approx 237.143$ correct to three decimal places.

Verification: We now verify our answer using a linear equation. The slope of the line connecting the points (0, 240) and (21, 230) is: $\dfrac{230-240}{21-0} = -\dfrac{10}{21}$.

Thus the equation of the line is,

$$y - 240 = -\frac{10}{21}(x - 0)$$

$$or \qquad y = -\frac{10}{21}x + 240$$

Now substituting $x = 6$ for the year *1960*, we obtain $y = -\dfrac{10}{21}(6) + 240$, which gives y = 237.143, the same answer as before.

We note that the midpoint formula is taught earlier in the curriculum as compared to equations of straight lines, and thus to be able to do the application presented above without using an equation of a straight line is noteworthy. Moreover, using the generalized formula with the ratio a : b, provides the answer quicker than using equation of a straight line, and give another way to obtain the result. However, the formula does have some drawbacks as compared to using straight lines, firstly it cannot find an x-value if y-value is given.

Also it *appears* that it cannot *extrapolate*, *e.g.*, it cannot find the y-value for the year 1990, in the application presented above. However it is not true because we can divide the segment connecting the points (x_1, y_1) and (x_2, y_2) externally in the ratio a:b to obtain the x and y coordinates as $\left(\dfrac{ax_2 - bx_1}{a-b}, \dfrac{ay_2 - by_1}{a-b}\right)$, where $a>b$.

Let us verify that it is in fact true. Since 90 is 36 units away from 54 and 15 units away from 75, therefore the ratio is 36:15, or 12:5. Thus using a = 12, and b = 5, in the above formula, we get $y = \dfrac{12(230) - 5(240)}{12-5}$ = 222.857.

The year 1990 is 36 units away from the year 1954. Thus substituting x = 36 in the equation of the *straight line* above, we obtain $y = -\frac{10}{21}(36) + 240$, This gives y = 222.857, the same value as we obtained above.

To summarize, this result generalizes the result that we have been teaching for a long time, and its proof (see Appendix A) shows that geometry and algebra are inter-connected. The Standards of NCTM have repeatedly emphasized to show the inter-connection between different branches of mathematics and also to include applications if and when possible. Finally, since this article is at the College Algebra level, it could provide for our students some fuel and motivation to pursue undergraduate research.

In the following appendix, we provide a proof of the formula to find the coordinates of a point that divides (internally) the segment connecting the points (x_1, y_1) and (x_2, y_2) in the ratio a : b, where 'a' and 'b', are any two real numbers. We also find the coordinates of the centroid of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

REFERENCES

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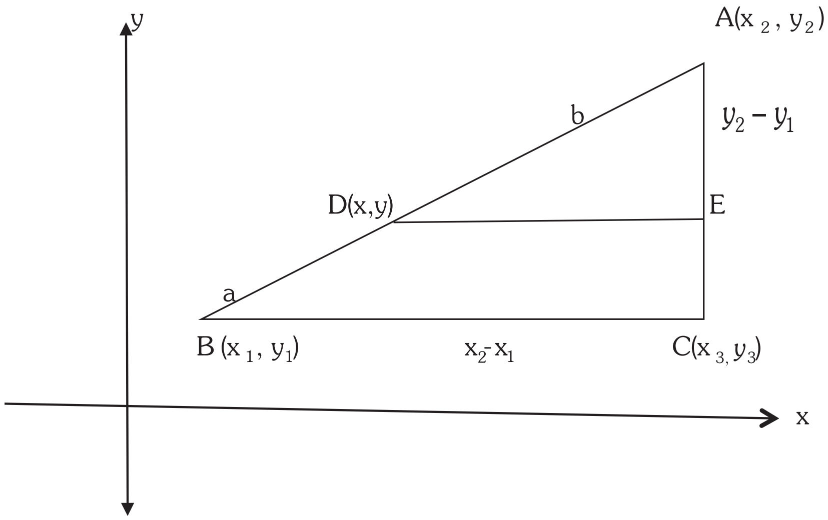
2. Problems In Plane and Solid Geometry, v.1 Plane Geometry, D. Efremov, Odessa, 1902, Victor Prasolov, p.16, Translated and edited by Dimitry Leites.

APPENDIX A

Theorem. Prove that the coordinates of a point dividing (internally) a segment connecting the points (x_1, y_1) and (x_2, y_2) in the ratio a : b, are $\left(\frac{ax_2 + bx_1}{a+b}, \frac{ay_2 + by_1}{a+b}\right)$ where a and b are any two real numbers.

In order to prove this result, we will use similar triangles shown in the following figure.



We note that the triangles ABC and ADE are *similar* by the angle-angle similarity, because angle A is common in both triangles and the angles E and C both being right angles are congruent. Thus, the corresponding sides of these two triangles are proportional, *i.e.*,

$$\frac{AC}{AE} = \frac{BC}{DE} = \frac{AB}{AD}$$

or $$\frac{y_2 - y_1}{y_2 - y} = \frac{x_2 - x_1}{x_2 - x} = \frac{a+b}{b} \qquad (1)$$

We now solve the equations in (1) above for x and y. Equating the first and the last expressions in equation (1) we get $\frac{y_2 - y_1}{y_2 - y} = \frac{a+b}{b}$

or $$\frac{by_2 - by_1}{a+b} = y_2 - y$$

or $$y = y_2 - \frac{by_2 - by_1}{a+b}$$

or $$y = \frac{ay_2 + by_2 - by_2 + by_1}{a+b}$$

or $$y = \frac{ay_2 + by_1}{a+b}.$$

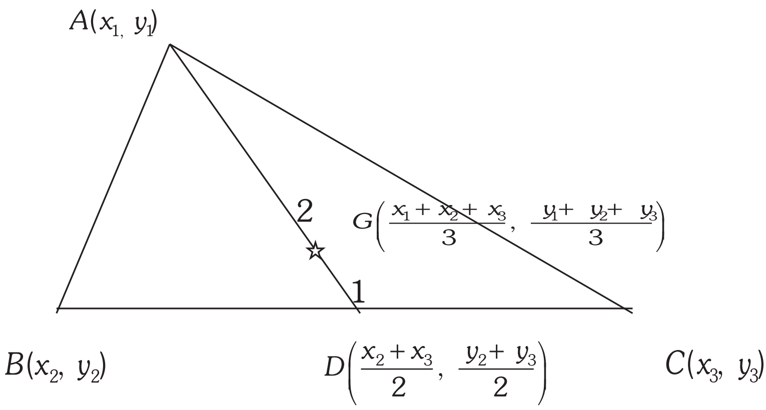
Similarly, equating the middle and the last expressions in equation (1) above, and simplifying for x, we obtain

$$x = \frac{ax_2 + bx_1}{a+b}.$$

Hence, the point dividing a segment connecting the points (x_1, y_1) and (x_2, y_2) in the ratio a : b, is $\left(\frac{ax_2 + bx_1}{a+b}, \frac{ay_2 + by_1}{a+b}\right)$, where a and b are any two real numbers.

The proof of showing that the coordinates of a point dividing (externally) a segment connecting the points (x_1, y_1) and (x_2, y_2) in the ratio a : b, are $\left(\frac{ax_2 - bx_1}{a-b}, \frac{ay_2 - by_1}{a-b}\right)$, where a, and b, are any two real numbers, with a > b is similar, and is left to a reader.

We next show that the co-ordinates of centroid of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the averages of x's and y's, as in the case of the midpoint.



First, we note that the co-ordinates of the midpoint D of the points $B(x_2, y_2)$ and $C(x_3, y_3)$ are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$. Since the centroid G divides the median AD in the ratio of 2:1, using the internal division formula as given above, the co-ordinates of the centroid G are: $\left(\frac{2\left(\frac{x_2 + x_3}{2}\right) + 1 \bullet x_1}{(2+1)}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + 1 \bullet y_1}{(2+1)}\right)$

which simplifies to $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, as desired.