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Properties Enjoyed by the Highest Digit in a Base Other than the Base 10

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PROPERTIES ENJOYED BY THE HIGHEST DIGIT IN A BASE OTHER THAN THE BASE 10

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ABSTRACT

The number nine in base ten enjoys some nice arithmetic properties. In this paper, we show that these properties are not intrinsic to the number nine; in fact, they are true for the largest digit in any base *b*. Four properties involving the final sums of all the digits of a number in a non-decimal base are explored and proofs of these properties are given in the appendix.

Key Words: Number theory, Place value, Non-decimal bases

INTRODUCTION

The number nine enjoys some nice arithmetic properties. In this paper, we show that the number nine is really not a mystifying number; in fact, the exact same arithmetic properties also hold true for the highest digit in any base different than the base 10. The key is the *highest digit* in a base. For example, the number 9 is the highest digit in the base 10 and 6 is the highest digit in the base 7.

After an extensive search of the literature, we could not find any discussion of the properties for the highest digit in any base *b* that we discuss below. Moreover, the properties for the number nine translate word by word for the highest digit in any base.

We first note that in this article, the phrase *the final sum of all the digits of a number* appears many times. Thus, we first present an example to explain what we mean by it. One of the properties of the number nine covered in an Early Childhood Education Majors' course, is that the final sum of all the digits in a nonzero multiple of the number nine always equals nine. For example, 22 • 9 = 198. The sum of the digits of the number 198 is 1 + 9 + 8 which equals 18, and *the final sum* of the digits is 1 + 8 or 9. Thus by *the final sum of all the digits of a number*, we mean that one should first add all the digits of the number and should continue to add all the digits of the subsequent numbers until one obtains a single digit number.

DISCUSSION

We will first list a few properties of the number nine. This set of properties of the number nine is not exhaustive. We then restate the same properties for the highest digit in any base and provide an example for each property to show that these properties are not intrinsic to the number nine. Finally, we prove these properties in Appendix A.

Property 1: Let x be a positive integer. If $x + 9 = y$, then the final sums of all the digits of x and all the digits of y are equal.
Example: If we add 376 and 9, we get 376 + 9 = 385. The final sums of all the digits of both the numbers 376 and 385 are equal. The sum for both is 7.

Property 2: Let x be a positive integer greater than nine. If $x - 9 = z$, then the final sums of all the digits of x and all the digits of z are equal.
Example: 356 – 9 = 347, and the final sums of all the digits of both the numbers 356 and 347 are equal. In fact, the sum is 5.

Property 3: The final sum of all the digits of a nonzero multiple of the number nine always equals nine.
Example: We have already presented an example (22 • 9 = 198) above demonstrating this property.

Property 4: Let x be a positive integer that is not a multiple of the number nine. If r is the remainder when we divide x by 9, then the remainder always equals the final sum of all the digits of the dividend x.
Example: The remainder of 188 ÷ 9 is 8. Also, the final sum of all the digits of the dividend 188 is 8.

A natural question arises: Why don't these properties hold for other digits in the base 10? Clues to the answer to this question resides in the fact that the number 9 is the highest digit in the base 10 number system; in fact, all of the above properties hold for the highest digit in **any** base b. For example, in base seven, the highest digit is 6, and $(5 \bullet 6)_7 = (42)_7$. Moreover, the sum of the digits 4 and 2 is $(4 + 2)_7 = (6)_7$. Consider another example: $(243 \bullet 6)_7 = (2154)_7$. The sum of the digits is $(2 + 1 + 5 + 4)_7 = (15)_7$, and finally, $(1 + 5)_7 = (6)_7$.

We now restate properties 1 – 4, for the largest digit in any base different than the base 10 and also provide an example corresponding to each property. We decided to provide the proofs of these properties in the Appendix A in order not to hinder the natural flow of this article.

NOTE: In order to keep the statements of the properties simple, we will denote the base as $b + 1$ instead of b and thus b will be our highest digit in the base $b + 1$.

In the statements of the properties 1' – 4' below, all the numbers x, y, z, r, and b are assumed to be the numbers in the base $b + 1$.

Property 1': Let x be a positive integer. If $x + b = y$, then the final sums of all the digits of x and of all the digits of y are equal in the base $b + 1$.

Property 2': Let x be a positive integer greater than b. If $x - b = z$, then the final sums of all the digits of x and of all the digits of z are equal in the base $b + 1$.

Property 3': The final sum of all the digits of a nonzero multiple of the number b always equals b in the base $b + 1$.

Property 4': Let x be a positive integer that is not a multiple of the number b. If r is the remainder when we divide x by b, then the remainder always equals the final sum of all the digits of the dividend x in the base $b + 1$.

In order to verify these properties and facilitate our discussion, we present an example for each of these properties in a non-decimal base. Since we used base 7 in our earlier discussion, we will also use it in the examples below.

The "base 7 arithmetic" is tedious and one needs to be very careful when verifying these properties. For property one, if we add $(665)_7$ and $(6)_7$, we obtain $(665)_7 + (6)_7 = (1004)_7$. The sum of the digits of $(665)_7$ is $(23)_7$, and the final sum is $(5)_7$. Also, the sum of the digits of $(1004)_7$ is $(5)_7$. This completes our example for the first property. To verify the second property, if we subtract $(6)_7$ from $(362)_7$, we obtain $(362)_7 - (6)_7 = (353)_7$. Again, the final sum of all the digits for both the numbers $(362)_7$ and $(353)_7$ is $(5)_7$. As an example for our third property, if we multiply $(245)_7$ with $(6)_7$, we obtain $(2202)_7$, and the sum of all digits of the number $(2202)_7$ is $(6)_7$. Finally, for an example of the fourth property, if we divide $(532)_7$ by $(6)_7$, the quotient is $(62)_7$ and the remainder is $(4)_7$. Moreover, the sum of all the digits of the dividend is $(13)_7$, and the final sum is $(4)_7$ which equals the remainder.

These examples verify that the conjectures made in the properties 1' – 4' above may be true. However, to be sure that the assertions made in properties 1' – 4' are in fact correct, we must prove these properties. We provide the proofs of these properties in the Appendix A.

To summarize, the number nine is not mystical or magical. The mystery about the number nine is that it is the highest digit in the base ten number system, and the special results about the number nine, in fact, hold for the highest digit in any base. The results presented in this paper about the highest digit in a base are not exhaustive; however, they represent a good sample and provide fuel for further explorations.

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APPENDIX A

In this appendix, we provide the proofs of the four properties discussed in the main body of the article. We thought that the proofs would be straight forward; however, they turned out to be involved.

We first note that properties one and two are in fact equivalent; in other words, 367 + 9 = 376 is equivalent to 376 – 9 = 367. Thus, we will only prove one of them. The proof of the first property follows.

Property 1': Let x be a positive integer. If $x + b = y$, then the final sums of all the digits of x and of all the digits of y are the same in the base $b + 1$.

Notation: Let S(x) denote the final sum of all the digits of x. Thus, we want to prove that in the base $b + 1$, S(x) = S(y).

Discussion: In order to provide motivation for the proof of this property, we first use the numbers in the base 10. Now S(10) = 1, thus S(x + 10) = S(x) + 1. Now we are given that $x + 9 = y$. Since $x + 9 = (x + 10) - 1$, therefore $(x + 10) - 1 = y$ which implies $x + 10 = y + 1$. Thus, S(x + 10) = S(y + 1). But S(y + 1) is S(y) + 1. So, S(y) + 1 = S(y + 1) = S(x + 10) = S(x) + 1. Thus, S(y) + 1 = S(x) + 1, and hence S(x) = S(y).

Proof: We must observe that in any base $b + 1$, the number $b + 1$ is equal to $(10)_{b+1}$, because $(10)_{b+1} = 1(b + 1)^1 + 0(b + 1)^0 = b + 1$. Thus adding $b + 1$ to a number x in the base $b + 1$ is equivalent to adding the number $(10)_{b+1}$. The hypothesis of the property is that in any base $b + 1$, $x + b = y$. If we represent $x + b$ as $(x + b +1) - 1$, then it follows $(x + b + 1) -1 = y$. This means that $(x + b +1) = y + 1$. But $(10)_{b+1} = b + 1$, so $(x + b + 1)$ is equal to $(x + (10)_{b+1})$. Thus, $(x + (10)_{b+1}) = y + 1$. Then $S(x + (10)_{b+1}) = S(y + 1)$. So it follows that $S(x) + S((10)_{b+1}) = S(y) + 1$. But $S((10)_{b+1}) = 1$ in base $b + 1$ so it follows that $S(x) + 1 = S(y) + 1$. We now conclude that $S(x) = S(y)$. Thus the final sum of the digits of x equals the final sum of the digits of y.

Property 3': The final sum of all the digits of a nonzero multiple of the number b always equals b in the base $b + 1$.

Proof: We prove the result by induction. Without loss of generality, we will only consider the positive multiples $b \bullet k$ of b. The proof for negative multiples is identical. We note that the result holds for $k = 1$. Since $b \bullet 1 = b$, in the base $b + 1$, the sum of the digits of $b \bullet 1$ equals the sum of the digits of b. Therefore, the result holds for $k = 1$.

For the inductive hypothesis, let us assume that the result holds for $k = n$, that is, in the base $b + 1$, the final sum of all the digits of the multiple $b \bullet n$ equals b. We next prove that the result holds for $k = n + 1$. We want to show that the final sum of all the digits of the multiple $b(n + 1)$ in the base $b + 1$ equals b. Since $(b \bullet (n + 1))_{b+1} = (b \bullet n)_{b+1} + b_{b+1}$, the final sum of all the digits in $b \bullet (n + 1)$ in the base $b + 1$ equals the final sum of all the digits in the multiple $b \bullet n$ plus the sum of all the digits in b in the base $b + 1$. By the inductive hypothesis, the final sum of all the digits of $b \bullet n$ equals b, in the base $b + 1$, therefore the final sum of all the digits in $b \bullet (n + 1)$ equals the final sum of all the digits in $b + b$ in the base $b + 1$. Rewriting $b + b$ as $(b + 1) + (b - 1)$ or $1 \bullet (b + 1)^1 + (b - 1)(b + 1)^0$, we have the digit 1 in the $(b + 1)^1$ place-value column and the digit $(b - 1)$ in the $(b + 1)^0$ place-value column. Hence the final sum of the digits in $b \bullet (n + 1)$ in the base $b + 1$, equals the sum of the digits of the numbers 1 and $b - 1$, that is, $1 + b - 1 = b$, as desired.

Property 4': Let x be a positive integer that is not a multiple of the number b. If r is the remainder when we divide x by b, then the remainder always equals the final sum of all the digits of the dividend x in the base $b + 1$.

NOTE: For the notational convenience in the proof of this property, we will use b as our base and $b - 1$ as the largest digit in the base b.

Proof: The crux of the proof lies in the fact that for any positive integer n, $(k - 1)$ always divides $(k^n - 1)$.

Let $x = d_{n+1} d_n d_{n-1} \ldots d_2 d_1$ be any $(n + 1)$ digit number in the base b, where $d_1, d_2, \ldots, d_{n+1}$ are the digits of the number x. We further assume that x is not a multiple of $(b - 1)$. We first show that dividing x by $(b - 1)$ is equivalent to dividing the sum of all the digits $d_{n+1} + d_n + \ldots + d_2 + d_1$, of x by $(b - 1)$. In the expanded form, the number x can be written in base b as:

$$x = d_{n+1} b^n + d_n b^{n-1} + \ldots + d_2 b^1 + d_1 b^0.$$

Rewriting x, by replacing each b^i with $(b^i + 1 - 1)$ for i from 1 to $n + 1$, we obtain,

$$x = d_{n+1}(b^n - 1 + 1) + d_n(b^{n-1} - 1 + 1) + \ldots + d_2(b^1 - 1 + 1) + d_1(b^0 - 1 + 1).$$

This can be further rewritten as:

$$x = d_{n+1}(b^n - 1) + d_n(b^{n-1} - 1) + \ldots + d_2(b^1 - 1) + d_1(b^0 - 1) + (d_{n+1} + d_n + \ldots + d_2 + d_1).$$

We can make the following conclusions from the last equation:

Since $b - 1$ divides each of the $b^i - 1$, for $0 \leq i \leq n$, (stated above as the crux of the proof), it follows that dividing x by $(b - 1)$ is equivalent to dividing the sum of all the digits $d_{n+1} + d_n + \ldots + d_2 + d_1$ of x by $(b - 1)$. In other words, the remainder obtained on dividing x by $(b - 1)$ is the same as the remainder obtained by dividing the sum of all the digits of x by $(b - 1)$.

We now have two cases:

Case 1: If the sum $d_{n+1} + d_n + \ldots + d_2 + d_1$, is less than $b - 1$ then it is itself the remainder and is obviously the sum of all the digits of x. Thus the result holds in this case.

Case 2: If the sum is greater than the divisor $(b - 1)$, then by continuing to divide this sum by $(b - 1)$, we would eventually obtain a single digit number, say $r < (b - 1)$, which is the remainder. Using the result from case 1 above and the fact obtained above in the proof, the remainder at each step will be the sum of all the digits of the dividend. Therefore, the remainder r will be the final sum of all the digits of x.