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The Mathematical Magic of Perfect Numbers

THE MATHEMATICAL MAGIC OF PERFECT NUMBERS

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ABSTRACT

Mathematicians have been fascinated for centuries by the properties and patterns of numbers [2]. They have noticed that some numbers are equal to the sum of all of their factors (not including the number itself). Such numbers are called perfect numbers. Thus a positive integer is called a perfect number if it is equal to the sum of its proper positive divisors. The search for perfect numbers began in ancient times. The four perfect numbers 6, 28, 496, and 8128 seem to have been known from ancient times [2]. In this paper, we will investigate some important properties of perfect numbers. We give easy and simple proofs of theorems such that students with calculus II skills can understand most of it. We give our own alternative proof of the well-known Euclid's Theorem (Theorem I). We will also prove some important theorems which play key roles in the mathematical theory of perfect numbers..

Key Words: Prime Numbers, Perfect numbers, and Triangular numbers.

INTRODUCTION AND BACKGROUND

Throughout history, there have been studies on perfect numbers. It is not known when perfect numbers were first studied and indeed the first studies may go back to the earliest times when numbers first aroused curiosity [3]. It is rather likely, although not completely certain, that the Egyptians would have come across such numbers naturally given the way their methods of calculation worked, where detailed justification for this idea is given [2]. Perfect numbers were studied by Pythagoras and his followers, more for their mystical properties than for their number theoretic properties [1]. Although, the four perfect numbers 6, 28, 496 and 8128 seem to have been known from ancient times and there is no record of these discoveries [5]. The first recorded mathematical result concerning perfect numbers which is known occurs in Euclid's *Elements* written around 300BC [3].

THE MAIN RESULTS

Proposition I: If 2^n-1 is prime, then n is prime for n>1.

Proof: Suppose *n* is not prime, then there exist positive integers a and b $n=axb, a>1, b>1$.

Then, $2^n-1=2^{ab}-1=(2^a-1)(2^{a(b-1)}+2^{a(b-2)}+2^{a(b-3)}\ldots+2^a+1)$

Since 2^n-1 is prime and $(2^{a(b-1)}+2^{a(b-2)}+2^{a(b-3)}\ldots+2^a+1)>1$, it follows that

$$2^a-1=1$$
$$\Rightarrow\ 2^a=2^1$$
$$\Rightarrow\ a=1$$

This is a contradiction, hence n is prime.

Remark I: The converse to proposition 1 is false as n = 11 is prime but 2^{11}-1 is not prime as 2^{11}-1 = 23x89.

Theorem I. If 2^k -1 (k>1) is prime, then n $=2^{k-1}$ $(2^k$-1) is a perfect number.

Proof: We will show that n = sum of its proper factors.

We will find all the proper factors of $2^{k-1}(2^k-1)$, and add them.

Since 2^k-1 is prime, let p = 2^k-1. Then n = p(2^k-1).

Let us list all factors of 2^{k-1} and other proper factors of n as follows.

Factors of 2^{k-1}	Other Proper Factors
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-2}p$

Adding the first column, we get:

$1+2+2^2+2^3\ldots+2^{k-3}+2^{k-2}+2^{k-1}$

$=2^k-1$

$=p$

Adding the second column, we get:

$p+2p+2^2p+2^3p\ldots+2^{k-4}p+2^{k-3}p+2^{k-2}p$

$=p(1+2+2^2+\ldots+2^{k-2})$

$=(2^{k-1}-1)p$

Now adding the two columns together, we get:
$p+p(2^{k-1}-1)$
$=p(1+2^{k-1}-1)$
$=p(2^{k-1})$
$=n$

Hence n is a perfect number.

Remark II: A question can be raised if k is prime by itself.

$\Rightarrow 2^{k-1}(2^k-1)$ is a perfect number. The answer is negative as it will be easily shown that it does not work for k=11.

Corollary I: If 2^k-1 is prime, $1+2+3+4\ldots+2^k-1$ is a perfect number.

Proof: Note that:

$$n=1+2+3+4\ldots+2^k-1$$
$$=\frac{(2^k-1+1)(2^k-1)}{2}$$
$$=\frac{2^k(2^k-1)}{2}$$
$$=2^{k-1}(2^k-1)$$

$\Rightarrow n$ is a perfect number by Theorem 1.

Corollary II: If 2^k-1 is prime, then $n=2^{k-1}+2^k+2^{k+1}\ldots+2^{2k-2}$ is a perfect number.

Proof: We have:

$$n=2^{k-1}+2^k+2^{k+1}\ldots+2^{2k-2}=2^{k-1}(1+2+2^2+2^3\ldots+2^{k-1})$$
$$n=2^{k-1}(2^k-1)$$

$\Rightarrow n$ is a perfect number by Theorem 1.

Remark III: Every even perfect number n is of the form $n = 2^{k-1}(2^k-1)$. We will not prove this, but we will accept and use it. So, the converse to Theorem 1 is also true. This is called Euler's Theorem.

Next we will show how Remark III applies to the first four perfect numbers. Note that:

$$6=2\bullet 3=2^1(2^2-1)=2^{2-1}(2^2-1)$$
$$28=4\bullet 7=2^2(2^3-1)=2^{3-1}(2^3-1)$$
$$496=16\bullet 31=2^4(2^5-1)=2^{5-1}(2^5-1)$$
$$8128=64\bullet 127=2^6(2^7-1)=2^{7-}1(2^7-1)$$

Theorem II. Every even perfect number n is a triangular number.

Proof: n is a perfect number⇒ n= $2^{k-1}(2^k-1)$ by Remark III.

Hence, n= $\frac{2^k(2^k-1)}{2} = \frac{(m+1)m}{2}$, where m=$2^k$-1. Thus n is a triangular number.

Corollary III. If T is a perfect number, then 8T +1 is a perfect square.

Proof: T is a perfect number ⇒ T is a triangular number.

⇒ T= $\frac{(m+1)m}{2}$ for some positive integer m.

$$\begin{aligned} \Rightarrow 8T+1 &= 4\,m(m+1) + 1 \\ &= 4m^2 + 4m + 1 \\ &= (2m+1)^2 \end{aligned}$$

Next we will prove two important theorems which play key roles in our study of perfect numbers .

Theorem III: The sum of the reciprocals of the factors of a perfect number n is equal to 2.

Proof: Let $n = 2^{k-1}(2^k-1)$ where $p = 2^k-1$ and is prime. Let us list all the possible factors of n.

Factors of 2^{k-1}	Other Factors
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-1}p$

Sum of reciprocals of factors of 2^{k-1}

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\ldots+\frac{1}{2^{k-1}}$$

$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}}{2\left(2^{k-1}\right)}+\frac{2^{k-1}}{2^2\left(2^{k-1}\right)}\ldots+\frac{1}{\left(2^{k-1}\right)}$$

$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}\ \ 2^{-1}}{2^{k-1}}+\frac{2^{k-1}\ \ 2^{-2}}{2^{k-1}}\ldots+\frac{1}{2^{k-1}}$$

$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-2}}{2^{k-1}}+\frac{2^{k-3}}{2^{k-1}}\ldots+\frac{1}{2^{k-1}}$$

$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}\ldots+1}{2^{k-1}}$$

$$=\frac{2^k-1}{2^{k-1}}=\frac{\mathrm{p}}{2^{k-1}}$$

Sum of reciprocals of other factors

$$\frac{1}{\mathrm{p}}+\frac{1}{2p}+\frac{1}{2^2p}+\frac{1}{2^3p}\ldots+\frac{1}{2^{k-1}p}$$

$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}}{2\left(2^{k-1}p\right)}+\frac{2^{k-1}}{2^2\left(2^{k-1}p\right)}\ldots+\frac{1}{\left(2^{k-1}p\right)}$$

$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}\ \ 2^{-1}}{2^{k-1}p}+\frac{2^{k-1}\ \ 2^{-2}}{2^{k-1}p}\ldots+\frac{1}{2^{k-1}p}$$

$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-2}}{2^{k-1}p}+\frac{2^{k-3}}{2^{k-1}p}\ldots+\frac{1}{2^{k-1}p}$$

$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}\ldots+1}{2^{k-1}p}$$

$$=\frac{2^k-1}{2^{k-1}p}=\frac{p}{2^{k-1}p}=\frac{1}{2^{k-1}}$$

Now the sums of reciprocals of all factors are equal to:

$$=\frac{p}{2^{k-1}}+\frac{1}{2^{k-1}}$$

$$=\frac{p+1}{2^{k-1}}$$

$$=\frac{2^k-1+1}{2^{k-1}}$$

$$=\frac{2^k}{2^{k-1}}=2$$

Corollary IV. No proper divisor of a perfect number can be perfect.

Proof: Suppose n is a perfect number and d its proper divisor. Let 1, x, x_2, $x_3 \ldots x_m$, d be divisors of d.

If d is a perfect number, then $1+\frac{1}{x}+\frac{1}{x_2}+\frac{1}{x_3}\ldots+\frac{1}{x_m}+\frac{1}{d}=2$, but this is not possible as 1, x, x_2, $x_3 \ldots x_m$, d are also factors of n and $1+\frac{1}{x}+\frac{1}{x_2}+\frac{1}{x_3}\ldots+\frac{1}{x_m}+\frac{1}{d}$ +(*sum of the reciprocals of other remaining factors*)=2. Hence, $1+\frac{1}{x}+\frac{1}{x_2}+\frac{1}{x_3}\ldots+\frac{1}{x_m}+\frac{1}{d}\neq 2$ and is not perfect.

To clearly understand Example 3, let us take n = 8128 and d = 508.

Factors of 508 are 1, 2, 4, 127, 254, and 508. Now observe that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{127}+\frac{1}{254}+\frac{1}{508}$$
$$=\frac{508+254+127+4+2+1}{508}$$
$$=\frac{896}{508}\neq 2$$

Factors of 8128 are: 1, 2, 4, 8, 16, 32, 64,127, 254, 508, 1016, 2032, 4064, and 8128. Note that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{127}+\frac{1}{254}+\frac{1}{508}+\frac{1}{1016}+\frac{1}{2032}+\frac{1}{4064}+\frac{1}{8128}$$
$$=\frac{8128+4064+2032+1026+508+254+127+64+32+16+8+4+2+1}{8128}$$
$$=\frac{16256}{8128}=2$$

Corollary V. No power of a prime can be a perfect number.

Proof. Let p be prime and let $n = p^k$. The factors of n are 1, p, p^2, $p^3 \ldots p^k$.

Now, we have:

$$1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}\dots+\frac{1}{p^k}$$

$$=1+\frac{p^{k-1}+p^{k-2}+p^{k-3}\dots+p+1}{p^k}$$

$$=1+\frac{p^k-1}{p^k(p-1)}$$

$$1+\frac{p^k-1}{p^k}$$

$$=1+\frac{p^k}{p^k}-\frac{1}{p^k}$$

$$=1+1-\frac{1}{p^k}$$

$$=2-\frac{1}{p^k}<2.$$

Therefore, n is not a perfect number.

Theorem IV. If n is a perfect number such that $n = 2^{k-1}(2^k-1)$, then the product of the positive divisor's of n is equal to n^k.

Proof: We list factors of n as in Theorem 2

Factors of 2^{k-1}	Other Factors
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-1}\ p$

Product of column 1 =

$$1^{*}\ 2^{*}\ 2^{2}\ {}^{*}\ 2^{3}\ldots{}^{*}\ 2^{k-1} = 2^{1+2+3\ldots+(k-1)} = 2^{\frac{k(k-1)}{2}}$$

Product of column 2 =

$$p \bullet 2p \bullet 2^{2}p\ldots\ 2^{k-1}p$$
$$= p^{k}(1 \bullet 2 \bullet 2^{2}\ldots 2^{k-1})$$
$$= p^{k}(2^{\frac{k(k-1)}{2}}),$$

Therefore the products of both columns are

$$= 2^{\frac{k(k-1)}{2}} \bullet p^{k} \bullet 2^{\frac{k(k-1)}{2}}$$
$$= 2^{k(k-1)} \bullet p^{k}$$
$$= (2^{k-1} \bullet p)^{k}$$
$$= \mathrm{n}^{k}.$$

Example 1: Apply Theorem IV to n = 28.

$n = 28 = 2^{2}(2^{3}\text{-}1)$ (Here $k = 3$).

Factors of 28 are 1, 2, 4, 7, 14, and 28.

The product of the factors of 28 =

$$1 \bullet 2 \bullet 4 \bullet 7 \bullet 14 \bullet 28$$
$$= 28 \bullet 28 \bullet 28$$
$$= 28^{3}$$

Example 2: Apply Theorem IV to n = 496.

$n = 496 = 2^{4}(2^{5}\text{-}1)$ (Here $k = 5$).

Factors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, and 496.

The products of the factors of 496 =

$$1 \bullet 2 \bullet 4 \bullet 8 \bullet 16 \bullet 31 \bullet 62 \bullet 124 \bullet 248 \bullet 496$$
$$= 496 \bullet 496 \bullet 496 \bullet 496 \bullet 496$$
$$= 496^{5}$$

Hunt for Some Perfect Numbers Using Calculator

$n=2^{k-1}(2^k-1)$ (2^k-1 is prime)

$p_1 = 2^1(2^2-1) = 6$ (k = 2)

$p_2 = 2^2(2^3-1) = 28$ (k = 3)

$p_3 = 2^4(2^5-1) = 496$ (k = 5)

$p_4 = 2^6(2^7-1) = 8128$ (k = 7)

$p_5 = 2^{12}(2^{13}-1) = 33550336$ (k = 13)

$p_6 = 2^{16}(2^{17}-1) = 8589869056$ (k = 17)

$p_7 = 2^{18}(2^{19}-1) = 137438691328$ (k = 19)

$p_8 = 2^{30}(2^{31}-1) = 2305843008139952128$ (k = 31)

$p_9 = 2^{60}(2^{61}-1) =$ 265845599156983174465469261595384
2176 (k = 61)

$p_{10} = 2^{88}(2^{89}-1) =$ 191561942608236107294793378084303
638130997321548169216 (k = 89)

$p_{11} = 2^{106}(2^{107}-1) =$ 131640364585696483372397534604587
22910223472318386943117783728128 (k = 107)

$p_{12} = 2^{126}(2^{127}-1) =$ 144740111546645244279463731260859884
81573677491474835889066354349131199152128 (k = 127)

$p_{13} = 2^{520}(2^{521}-1) =$ 235627234572673470657895489967099
04988477547858392600710143027597506337283178622
23973036553960260056136025556646250327017505289
25780432155433824984287771524270103944969086640
28644534128033831439790236838624033171435922356643
21970310172071316352748729874740064780193958716
593640108741937564905791854949216055564697 6
(k = 521)

$p_{14} = 2^{606}(2^{607}-1) =$ 141053783706712069063207958086
063189881486743514715667838838675999954867742652
38011410419332903769025156195056870982932716408
77243663700871167312681593136524874506524398058772
962072974467232951666582288469268077866528701889

20867879451478364569313922060370695064736073572378695176473055266826253284886383715072974324463835300053138429460296575143368065570759537328128

(k = 607)

CONCLUSION

We try our best to make this paper very interesting and attractive to many readers of mathematics and other similar fields. The proofs of theorems and other results are given in such a way that somebody with basic number theory skills can easily read and understand them. We strongly believe that the mathematics of perfect numbers is rich in research problems for mathematicians to work on in the future.

OPEN QUESTIONS [4]

We were able to observe that there are open questions concerning perfect numbers which can be excellent potential research problems.. The following are the open questions which are potential research problems to work on.

1. Are there any odd perfect numbers or are all perfect numbers even?
2. Is there a finite number of perfect numbers or are there infinitely many?

We hope that sometime in the near future these questions will be answered as the famous Fermat's Last Theorem did.

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