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OPTIMIZATION OF A BALL'S LAUNCH IN SPORTS

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ABSTRACT

Newton's laws are used to study the effects of air resistance on an object's motion. In ball-related sports such as baseball, soccer, etc., understanding the effects of air resistance is essential to optimize ball launch performance. This performance optimization can be studied by identifying the minimal time it takes for a ball with speed v to travel a certain distance. We work with two models that apply to an object's motion. One of the models assumes a linear air drag while a second model makes use of a quadratic air drag. We do investigate known differential equations for when the Magnus force is present as well as absent. This is done through numerical and analytic solutions, when possible. The development of approximations leads to differential equations that are suitable for time optimization studies The analytic calculations are compared to MATLAB's numerical results. We concentrate on situations for which the speed of the projectile parallel to the ground is much greater than its speed perpendicular to it.

***Keywords*:** Mechanics, speed, sports, motion, air drag

INTRODUCTION

The use of technology in sports allows for a deeper understanding of the physics of a sport, leading to increased knowledge and skill of play. In ball-related sports, such as baseball, soccer and football, understanding the physics behind projectile motion is necessary to optimize performance. Analyzing the effect of air drag is one method allows for the optimization of play. Generally, air drag is expressed as a force that acts to oppose the velocity of the object. The understanding of the effects of air drag on a ball's travel characteristics enables us to use this knowledge to allocate players in specific positions in order to effect an optimal outcome. The optimization in a ball's launch in sports can be identified through the decrease of flight time over a distance. In this paper, we consider functions that identify the minimum flight time of a ball between two distances. In a linear air drag force model the force is proportional to the negative of the velocity, while in the quadratic model the drag force is proportional to the product of the speed squared and the velocity unit vector.

In the work of Goff and Carre (Goff and Carre 2009), the authors discussed the trajectory analysis of a soccer ball. In particular they studied the motion of the ball in the presence of gravity, drag, and the Magnus force. The drag force on a sports projectile has been studied (Mestre 1990) and under low speeds it can be proportional to the speed, and in many problems the form of the drag force can be other than linear, for example, quadratic, etc. In our work here we consider a linear and a quadratic behavior, also, the drag force always points in a direction so as to oppose the projectile's velocity direction. The Magnus force, which is a force that arises due to the projectile's spin while moving through the air has two components. One component produces a lift force;

that is, it points perpendicularly to the drag force and remains in the plane formed by the velocity unit vector ($\hat{v}$) and the acceleration due to gravity ($\hat{j}$). Its direction is indicated by the symbol $\hat{\ell}$. The other component of the Magnus force is a sideways force whose direction is perpendicular to both $\hat{\ell}$ and $\hat{v}$. It should be mentioned that the Magnus force effect is a particular manifestation of Bernoulli's theorem: fluid pressure decreases at points where the speed of the fluid increases. In the case of a ball spinning through the air, the turning ball drags some of the air around with it (Magnus-Effect).

METHODS

1.A General Formulation

According to ref. [1] Newton's equation of motion associated with a ball traveling through the air is written as

$$m\bar{a} = \bar{F}_D + \bar{F}_L + \bar{F}_S + m\bar{g}, \qquad (1)$$

where $\bar{F}_D$ is the force due to drag, $\bar{F}_L$ is the lift part of the Magnus force, $\bar{F}_S$ is the sideways part of the Magnus force, and $m\bar{g}$ is the force due to the projectile's weight, with m as the mass of the ball. We note that the forces $\bar{F}_D$, $\bar{F}_L$, and $\bar{F}_S$ have a dependence on speed as determined by the drag force. Their main difference however, as mentioned above, lies in their direction[1,2]. In a standard (x,y,z) coordinate system, while the force due to gravity points in the negative y direction, $\bar{g} = -g\hat{j}$, for the case of a drag force that involves a quadratic behavior in speed, the rest of the forces are written as

$$\bar{F}_D = -\gamma_D v^2 \hat{v},$$

$$\bar{F}_L = \gamma_L v^2 \hat{\ell},$$

and

$$\bar{F}_S = \gamma_s v^2 (\hat{\ell} \times \hat{v})$$

with the various γ_i's being the coefficients of proportionality for each i=D, L, and S force cases. Here, v is the magnitude of the ball's speed, $\hat{v} = \frac{v_x}{v}\hat{i} + \frac{v_y}{v}\hat{j}$; that is, in the present case, we consider the (x,y) plane of motion alone; and finally, $\hat{\ell} = -\frac{v_y}{v}\hat{i} + \frac{v_x}{v}\hat{j}$; i.e., perpendicular to $\hat{v}$, as stated previously. Since we concentrate on a planar motion, we ignore $\bar{F}_S$ and with the above understanding write our full equations of motion as follows

$$m\bar{a} = ma_x\hat{i} + ma_y\hat{j} = -\gamma_D v\left(v_x\hat{i} + v_y\hat{j}\right) + \gamma_L v\left(-v_y\hat{i} + v_x\hat{j}\right) - mg\hat{j}. \qquad (2)$$

This means that, since $\vec{a} = d\vec{v}/dt$, for the case of a quadratic drag behavior, we have the coupled system of equations for v_x and v_y given by

$$m\frac{dv_x}{dt} = -\gamma_D v_x\sqrt{v_x^2+v_y^2} - \gamma_L v_y\sqrt{v_x^2+v_y^2}\,, \tag{3a}$$

and

$$m\frac{dv_y}{dt} = -\gamma_D v_y\sqrt{v_x^2+v_y^2} + \gamma_L v_x\sqrt{v_x^2+v_y^2} - mg\,, \tag{3b}$$

where, for the magnitude of the velocity we've used $v = \sqrt{v_x^2+v_y^2}$. In a similar fashion, if the drag force involves a linear behavior in speed, the forces considered here are now

$$\vec{F}_D = -C_D v\hat{v}\,,$$

and

$$\vec{F}_L = C_L v\hat{\ell}\,,$$

which now involve coefficients C_D and C_L for unit consistency. Following a similar process as for obtaining equations (3), in this linear case the resulting coupled equations (linear coupled approximation or LCA) of motion are

$$m\frac{dv_x}{dt} = -C_D v_x - C_L v_y\,, \tag{4a}$$

and

$$m\frac{dv_y}{dt} = -C_D v_y + C_L v_x - mg\,. \tag{4b}$$

It is worth noting that both equations (3b) and (4b) for the y motion each involves a term that counters the acceleration due to gravity, such terms are due to the Magnus effect and lengthen the time a projectile stays aloft.

In most sports, the general goal of a competition is to hurl a ball with as great amount of speed in the x direction as possible. This being the case in our analysis here, we see that equations (3) can be simplified if we let $v_x >> v_y$ which leads to the approximation $\sqrt{v_x^2+v_y^2} \approx v_x$ for the quadratic speed drag case and to replace our equations (3) with

$$m\frac{dv_x}{dt} = -\gamma_D v_x^2 - \gamma_L v_x v_y\,, \tag{5a}$$

and

$$m\frac{dv_y}{dt} = -\gamma_D v_x v_y + \gamma_L v_x^2 - mg\,, \tag{5b}$$

which here are referred to as the (quadratic coupled approximation or QCA) equations of motion. Thus, the set of equations (4) and (5) pertain to our analysis here as follows below.

1.B Main Approximations

In this subsection, we make the decoupled approximations; that is, we modify the (LCA) equations (4) by ignoring the coupling between the variables v_x and v_y we then have the linear equations of motion (linear decoupled approximation or LDA) as

$$m\frac{dv_x}{dt} = -C_D v_x \, , \tag{6a}$$

and

$$m\frac{dv_y}{dt} = -C_D v_y - mg \, . \tag{6b}$$

If we do the same for the QCA equations (5), we obtain the corresponding quadratic equations of motion (quadratic partially decoupled approximation or QPDA) are

$$m\frac{dv_x}{dt} = -\gamma_D v_x^2 \, , \tag{7a}$$

and

$$m\frac{dv_y}{dt} = -\gamma_D v_x v_y - mg \, . \tag{7b}$$

To assess the significance of these approximations, it is important that we discuss the solutions of the various differential equations presented above; that is, the LCA and QCA equations (4, 5) as well as the LDA and QPDA equations (6, 7). We do this in the following subsection.

1.C Analytic Solutions

In this subsection we write down analytic solutions (when possible) for the various cases discussed in subsections 1.A and 1.B. In the case of the (LCA), we work with equations (4), we have for $v_x(t)$ and $v_y(t)$,

$$v_x(t) = \frac{e^{-\frac{C_D \mathrm{t}}{m}}\left(mgC_L \mathrm{e}^{\frac{C_D \mathrm{t}}{\mathrm{m}}} + \cos\left(\frac{C_L \mathrm{t}}{\mathrm{m}}\right)\left(v_{\mathrm{xo}}\left(C_D{}^2 + C_L{}^2\right) - mgC_L\right) - \sin\left(\frac{C_L \mathrm{t}}{\mathrm{m}}\right)\left(mgC_D + v_{\mathrm{yo}}\left(C_D{}^2 + C_L{}^2\right)\right)\right)}{C_D{}^2 + C_L{}^2},$$

$$v_y(t) = \frac{e^{-\frac{C_D \mathrm{t}}{m}}\left(mgC_D\left(-\mathrm{e}^{\frac{C_D \mathrm{t}}{\mathrm{m}}}\right) + \sin\left(\frac{C_L \mathrm{t}}{\mathrm{m}}\right)\left(v_{\mathrm{xo}}\left(C_D{}^2 + C_L{}^2\right) - mgC_L\right) + \cos\left(\frac{C_L \mathrm{t}}{\mathrm{m}}\right)\left(mgC_D + v_{\mathrm{yo}}\left(C_D{}^2 + C_L{}^2\right)\right)\right)}{C_D{}^2 + C_L{}^2},$$

(8a)

with initial speed values $\mathrm{v_{xo}} . \mathrm{v_{yo}}$ at t=0, respectively. The related $x(t) = \int v_x(t)\, dt$, $y(t) = \int v_y(t)\, dt$ results are

$$x(t)=\frac{e^{-\frac{C_D t}{m}}}{\left(C_D{}^2+C_L{}^2\right)^2}\left(\begin{array}{l} e^{\frac{C_D t}{m}}\left(-2C_D C_L \mathrm{gm}^2+\mathrm{m}\left(C_D{}^2+C_L{}^2\right)(C_L \mathrm{gt}+C_D v_{xo}\text{-}C_L v_{yo})+x_o\left(C_D{}^2+C_L{}^2\right)^2\right)+ \\ m\sin\left(\frac{C_L \mathrm{t}}{m}\right)\left(gm\left(C_D{}^2\text{-}C_L{}^2\right)+\left(C_D{}^2+C_L{}^2\right)(C_L v_{xo}+C_D v_{yo})\right)+ \\ m\cos\left(\frac{C_L \mathrm{t}}{m}\right)\left(C_L\left(2C_D \mathrm{gm}+v_{yo}\left(C_D{}^2+C_L{}^2\right)\right)\text{-}C_D v_{xo}\left(C_D{}^2+C_L{}^2\right)\right) \end{array}\right)$$

,

$$y(t)=\frac{e^{-\frac{C_D t}{m}}}{\left(C_D{}^2+C_L{}^2\right)^2}\left(\begin{array}{l} m\sin\left(\frac{C_L \mathrm{t}}{m}\right)\left(C_L\left(2C_D \mathrm{gm}+v_{yo}\left(C_D{}^2+C_L{}^2\right)\right)\text{-}C_D v_{xo}\left(C_D{}^2+C_L{}^2\right)\right)- \\ m\cos\left(\frac{C_L \mathrm{t}}{m}\right)\left(gm\left(C_D{}^2\text{-}C_L{}^2\right)+\left(C_D{}^2+C_L{}^2\right)(C_L v_{xo}+C_D v_{yo})\right)+ \\ e^{\frac{C_D t}{m}}\left(\begin{array}{l} gm\left(m\left(C_D{}^2\text{-}C_L{}^2\right)-C_D \mathrm{t}\left(C_D{}^2+C_L{}^2\right)\right)+ \\ \left(C_D{}^2+C_L{}^2\right)\left(m(C_L v_{xo}+C_D v_{yo})+y_o\left(C_D{}^2+C_L{}^2\right)\right) \end{array}\right) \end{array}\right),$$

(8b)

with respective initial values x_o, y_o at t=0.

For the case of the (QCA) of equations (5), no analytic solutions were obtained and, later, whatsoever calculations we perform here related to the QCA, we do so numerically. For the LDA equations (6) we have for the velocity component solutions,

$$v_x(t)=v_{xo}e^{-\frac{C_D t}{m}},$$

$$v_y(t)=\frac{1}{C_D}e^{-\frac{C_D t}{m}}\left(C_D v_{yo}-mg\left(e^{\frac{C_D t}{m}}-1\right)\right),$$

(9a)

and corresponding $x(t)$, $y(t)$ components written as

$$x(t)=\frac{1}{C_D}\left[-mv_{xo}e^{-\frac{C_D t}{m}}+C_D \mathrm{x}_o+mv_{xo}\right],$$

$$y(t)=\frac{1}{C_D^2}\left[mg\left(-me^{-\frac{C_D t}{m}}-C_D t+m\right)+C_D\left(m\left(v_{yo}-v_{yo}e^{-\frac{C_D t}{m}}\right)+C_D \mathrm{y}_o\right)\right],$$

(9b)

where v_{xo} . v_{yo}, x_o, y_o as before.

Similarly, we can write the results for the QPDA of equations (7) as

$$v_x(t) = \frac{m\mathrm{v}_{xo}}{m + \gamma_D \mathrm{v}_{xo} t},$$

$$v_y(t) = \frac{2mv_{yo} - gt(2m + \gamma_D v_{xo} t)}{2(m + \gamma_D v_{xo} t)},$$

(10a)

with the associated $x(t)$, $y(t)$ components given by

$$x(t) = \frac{1}{\gamma_D}\left[m\log\left(\frac{m}{v_{xo}} + \gamma_D t\right) - m\log\left(\frac{m}{v_{xo}}\right) + \gamma_D \mathrm{x}_o \right],$$

$$y(t) = \frac{1}{4\gamma_D{}^2 v_{xo}^2}\left[\begin{array}{l} 2m(mg + 2\gamma_D v_{xo} v_{yo})\log\left(\frac{m}{v_{xo}} + \gamma_D t\right) - \gamma_D v_{xo}(g(2m + \gamma_D v_{xo} t)t + 4\gamma_D v_{xo} \mathrm{y}_o) \\ -2m\log\left(\frac{m}{\mathrm{v}_{xo}}\right)(mg + 2\gamma_D v_{xo} v_{yo}) \end{array} \right],$$

(10b)

and with initial values at t=0 as before. Also, the discussion of each of the analytic approximations and their comparison to the associated numerical solutions follows in the next subsection.

1.D Comparison among Solutions

In this subsection, a comparison is made between the various approximations presented above; that is, the QCA, QPDA, LCA, and QDA. The purpose of this comparison is to observe the differences and/or similarities of the various approximations under the present conditions. We start by solving equations (5) for the QCA, which is the most complicated set of coupled equations studied here. No analytic solutions were found and only their numerical results are shown by the dots in Figure 1's left panels for the x(t) and y(t) solutions. The numerical solutions were carried out by the Runge-Kutta (2,3) pair of Bogacki and Shampine method (Bogacki and Shampine 1989) which is used by the MATLAB (MATLAB) function ode23 (ODE23). More information as to the numerical solutions of differential equations can be found (Hasbun 2008). The derivations, however, of the analytic equations in subsection 1.C have been carried out using Mathematica (MATHEMATICA). In a similar way we obtained the numerical solutions to the QPDA, equations (7), as shown by the dots in Figure 1's right panels. Since we have analytic solutions for this system, in equations (10), they are shown by the superimposed circles. The agreement between the analytic and numerical solutions give confidence in the accuracy of the results. While the equations are very different, we notice that decoupling the x and y components of the differential equations does not change very much the x(t) behavior in the solutions. The y(t) motion is different because equations (5) have the extra term in $\dot{v}_y(t)$; i.e., the

Magnus force contribution, which tends to keep the projectile aloft for longer time, as mentioned before. In the QPDA, that term has been taken out and the observed difference is not surprising. However, in our present study, we are more concerned with the x(t) behavior which is shown to be described reasonably well with equation (7a) of the QPDA.

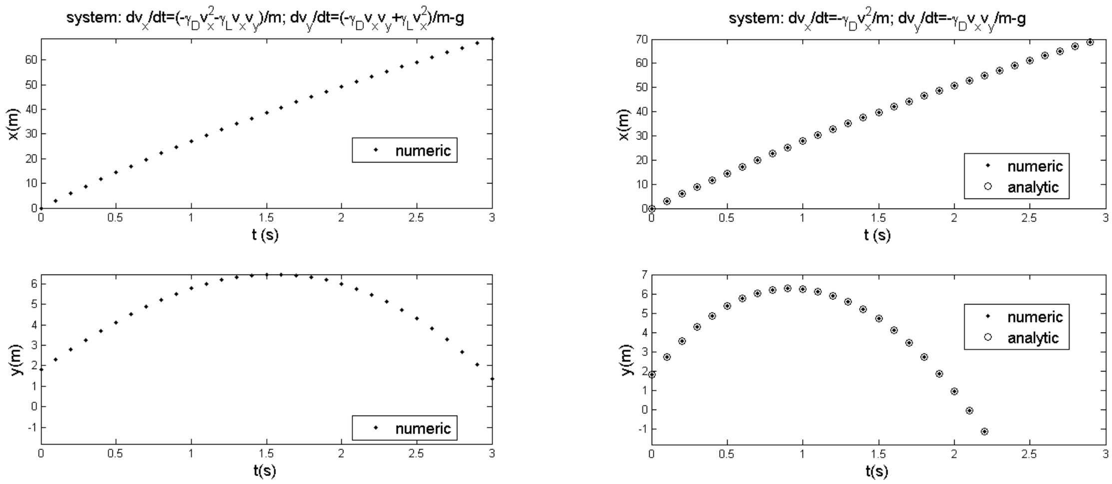

Figure 1. The results of the solutions to the QCA equations (5) [left panels] numerically (dots) and to the QPDA equations (7) [right panels] numerically (dots) and its analytic solutions (circles) of equations (10). The solutions shown are for the x(t) and y(t) components in meters and time in seconds.

Proceeding to the linear cases of equations (4) for the LCA and equations (6) for the LDA, their results are shown in Figure 2. The left panel are the LCA numerical results (dots) using the method mentioned above (Bogacki and Shampine 1989) and the circles are the analytic results of equations (8). The right panels are the LDA numerical results (dots) and the circles represent the results of the analytic solutions of equations (9). First, we notice that the numerical and analytic results for both approximations compare very well with one another for both approximations. Furthermore, the x(t) and y(t) curves for both approximations compare well with one another. The reason for this is that the coupling is not as important for the linear case in addition to the fact that in our situation the drag coefficient used is small. In the calculations carried out here in Figures 1 and 2, the parameters used are as follows: $m = 0.14 kg$, $\gamma_D = 0.001 kg/m$, $\gamma_L = 0.015 kg/m$, $C_D = 0.001 kg/s$, $C_L = 0.015 kg/s$, $v_{xo} = 30 m/s$, $v_{yo} = 5 m/s$, $x_o = 0 m$, and $y_o = 1.8 m$ for the initial values of x, y speed directions as well as x,y positions, respectively.

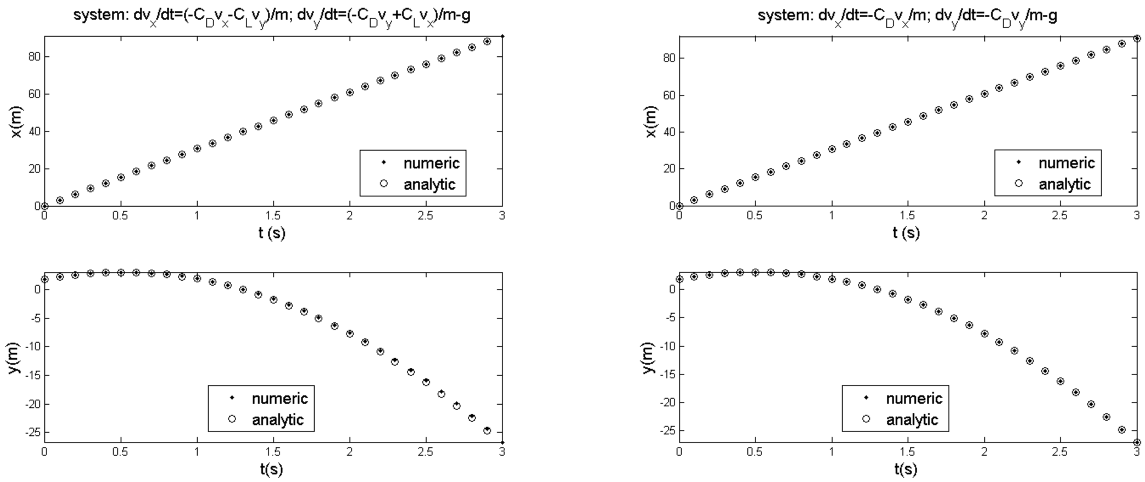

Figure 2. The left panels show the results of the solutions to the LCA equations (4) numerically (dots) and its analytic results (circles) of equations (8). The right panels show the numerical solutions to the LDA equations (6) (dots) compared to the analytic solutions (circles) equations (9). The solutions shown are for the x(t) and y(t) components in meters and time in seconds.

2.A Application Models

In keeping with the goal of this work, and in the spirit of the previous section, we look at the decoupled cases of the linear and the quadratic models above with the goal of optimizing the position of a relayed player (relayee) to whom a projectile (ball) is passed by an initial player (relayer) farther away. The concept is to find the best position of the relayee in order to minimize the time for the projectile to reach a final destination once the relayee disposes of the projectile toward its final destination. In the case of baseball, considered here as an example, the relayer is an outfielder, the relayee is the shortstop, and the final destination is home plate. The equations presented above are already involved and the decoupled cases are much easier to analyze. Also, given the favorable comparison between the coupled and decoupled equations of the previous section, and given the fact that the y-motion is negligibly affected in the cases for which the speed in the x-direction is the main factor in the motion, we consider two models. Both obtained from the above section. The first model is a linear one, which, from equation (6a), we have,

$$m\frac{dv}{dt} = -cv, \tag{11}$$

where since in this analysis we only work in one dimension, we've replaced v_x with v and also we are using simply c rather than C_D. In the same spirit, the second model is the quadratic one of equation (7a) which we write here as

$$m\frac{dv}{dt} = -\gamma v^2, \tag{12}$$

where we use γ rather than γ_D for simplicity. As before, we keep in mind that γ for the quadratic model and c for the linear one use different units; that is, kg/m for the quadratic case and kg/s for the linear case. As can be seen, due to the process involved in the analysis we are undertaking, equations (11) and (12) are simplified versions of the full Newtons's equations presented in the previous section.

2.B The Linear Model

Since the solutions to equation (11) are found in equations (9), we rewrite

$$v(t) = v_0 e^{-\frac{c}{m}t}, \tag{13a}$$

and

$$x(t) = x_0 + \frac{m}{c} v_0 \left(1 - e^{-\frac{c}{m}t}\right). \tag{13b}$$

According to equation (13b), starting from an initial position x_0 at a speed of v_0 the time it takes for the projectile to reach a final distance x is thus

$$t = -\frac{m}{c} \ln\left[\frac{mv_0 - c(x - x_0)}{mv_0}\right]. \tag{14}$$

We can now specialize to the baseball scenario alluded to previously. The outfielder (player 1) is located at L and the shortstop (player 2) is at L-x. Player 1 throws the ball at a velocity of v_{01} to player 2, who in turn throws the ball at a velocity of v_{02} toward home plate. Here we also assume that there is a time delay, Δt, associated with player 2's throw. Thus the total time it takes for the ball to reach home plate is identified as,

T(x) = time it takes player 1's throw to reach shortstop + delay time Δt + time it takes player 2's throw to reach home plate.

(15)

This scenario is illustrated in Figure 3, where we identify $x_1 - x_{01} = L$ to be the location of player 1 and $x_2 - x_{02} = x$ to be the location of player 2.

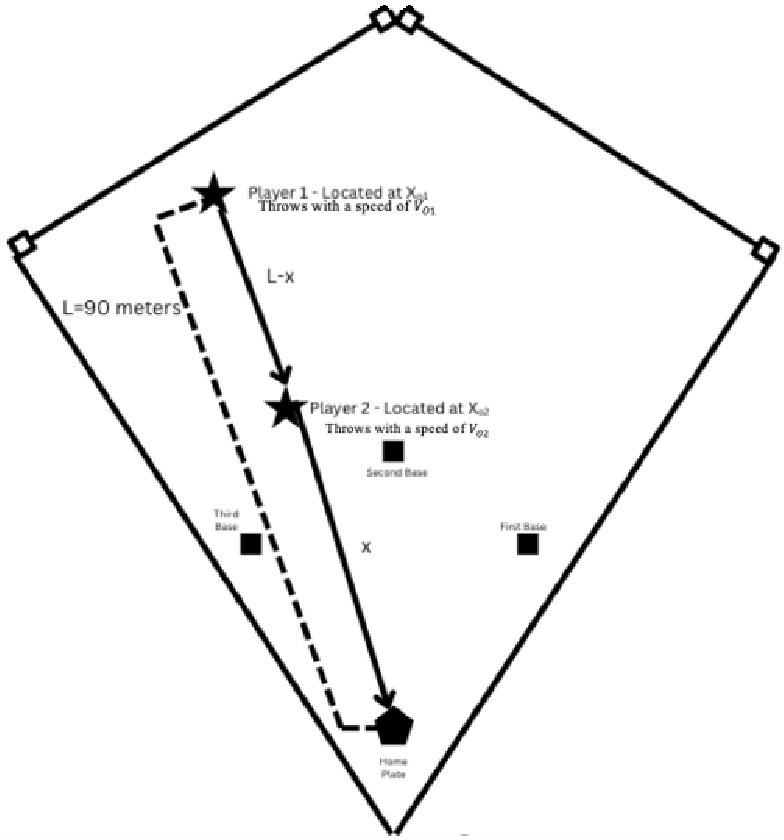

Figure 3. Diagram of certain baseball scenario in which player 1 throws the ball at a distance L-x to player 2, who then turns around and throws the ball distance x to home plate. The parameters for this scenario are as follows: $x_1 - x_{01} = L - x$, $L = 90m$, $x_2 - x_{02} = x$, $v_{01} 30.67 m/s$, $v_{02} = 35.28 m/s$, $c = 0.01 kg/s$, $m = 0.14 kg$, and $\Delta t = 0.5s$.

With the help of the general formula (14), equation (15) becomes a template to the situation of interest and leads to

$$T(x) = -\frac{m}{c}\ln\left[\frac{mv_{01} - c(L-x)}{mv_{01}}\right] + \Delta t - \frac{m}{c}\ln\left[\frac{mv_{02} - cx}{mv_{02}}\right], \qquad (16)$$

where we are letting $x_1 - x_{01} = L - x$, $L = 90m$, $x_2 - x_{02} = x$, $v_{01} 30.67 m/s$, $v_{02} = 35.28 m/s$, $c = 0.01 kg/s$, $m = 0.14 kg$, and $\Delta t = 0.5s$. Our interest lies in optimizing the total ball's flight time, so that using equation (16) and setting it's derivative with respect to x equal to zero; that is,

$$T'(x) = -m\left[\frac{1}{mv_{01} - c(L-x)} + \frac{1}{mv_{02} - cx}\right] = 0\,, \tag{17}$$

and yields the position of player 2 that minimizes the ball's travel time, or

$$x_{\min} = \frac{1}{2}\left[L + \frac{(v_{02} - v_{01})}{(c/m)}\right]. \tag{18}$$

Substituting this $x_{\min}$ back into equation (16) yields the ball's minimum travel time or,

$$T(x_{\min}) = -\frac{m}{c}\left(\ln\left[\frac{m\left(\frac{v_{01}+v_{02}}{2}\right) - \frac{cL}{2}}{mv_{01}}\right] + \ln\left[\frac{m\left(\frac{v_{01}+v_{02}}{2}\right) - \frac{cL}{2}}{mv_{02}}\right]\right) + \Delta t\,. \tag{19}$$

As mentioned before, the speed in the x-direction is the main factor in the motion, and that's what's considered here, so that for testing the accuracy of these results, Equation (16) is plotted as a function of x as shown in Figure 4. Notice that T(x) reaches a minimum at x=77.0286 meters which is in agreement with what equation (18) obtains for the same parameters; additionally, $T(x_{\min})$ of equation (19) obtains 3.3033 seconds which is what the plot of equation (16) shows as its minimum value in Figure 4 as well. Appendix A contains the script used to create Figure 4 for the purpose of the reader who is interested in experimenting with various parameters using this approximation.

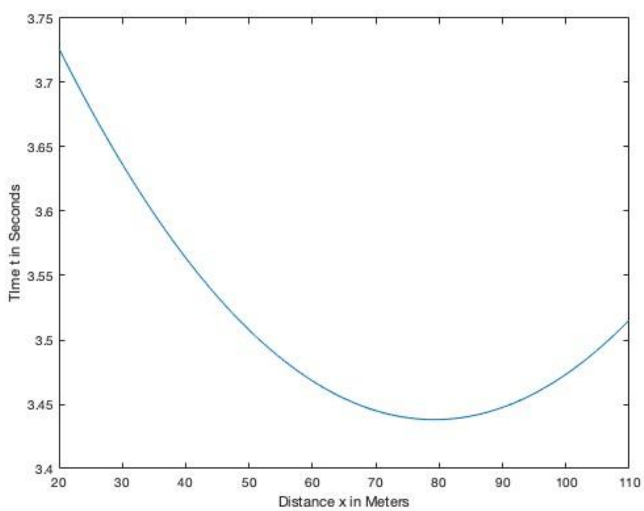

Figure 4. Results of $T(x_{\min})$ for Model 1 produced by MATLAB. Shows a minimum at x=77.2086 meters at t=3.3033 seconds. The parameters for used are: $x_1 - x_{01} = L - x$, $L = 90m$, $x_2 - x_{02} = x$, $v_{01} = 30.67m/s$, $v_{02} = 35.28m/s$, $c = 0.01kg/s$, $m = 0.14kg$, and $\Delta t = 0.5s$.

2.B The Quadratic Model

In a similar way to what we did in the above subsection, but now working with equation (12) whose solutions are given by equations (10), which we rewrite as

$$v(t)=\frac{v_0}{1+\frac{\gamma}{m}v_0 t}, \qquad (20a)$$

and

$$x(t)=x_0+\frac{m}{\gamma}\ln\left(1+\frac{v_0\gamma}{m}t\right). \qquad (20b)$$

where, again, from (20b) starting from an initial position x_0 at a speed of v_0 the time it takes for the projectile to reach a final distance x, for this model is

$$t(x)=\frac{m}{\gamma v_0}\left(e^{\frac{\gamma}{m}(x-x_0)}-1\right). \qquad (21)$$

Using this equation along with the understanding of equation (15), we arrive at the projectile's flight time for this model as

$$T(x)=\frac{m}{\gamma v_{01}}\left(e^{\frac{\gamma}{m}(L-x)}-1\right)+\Delta t+\frac{m}{\gamma v_{02}}\left(e^{\frac{\gamma}{m}x}-1\right), \qquad (22)$$

whose derivative, as before, is set to zero, to obtain the value of x that minimizes this time or

$$x_{\min}=\frac{L}{2}+\frac{m}{2\gamma}\ln\left(\frac{v_{02}}{v_{01}}\right), \qquad (23)$$

which, when substituted back into equation (22) gives the minimum time,

$$T(x_{\min})=\frac{m}{\gamma}\left(\frac{e^{-\frac{\gamma}{2m}\left(\frac{m}{\gamma}\ln\left(\frac{v_{02}}{v_{01}}\right)-L\right)}-1}{v_{01}}+\frac{e^{\frac{\gamma}{2m}\left(\frac{m}{\gamma}\ln\left(\frac{v_{02}}{v_{01}}\right)+L\right)}-1}{v_{02}}\right)+\Delta t\,. \qquad (24)$$

Again, the speed in the x-direction being the main factor in our approximation here, as in Model 1, equation (22) can be plotted as a function of x as shown in Figure 5. Notice that T(x) reaches a minimum at x=45.9783 meters which is in agreement with what equation (23) obtains for the same parameters. Also, $T(x_{\min})$ =20.8290 seconds, which is what equation (24) obtains. Similar to the linear model, Appendix B contains the script used to create Figure5 for the purpose of the reader who is interested in experimenting with various parameters using this approximation.

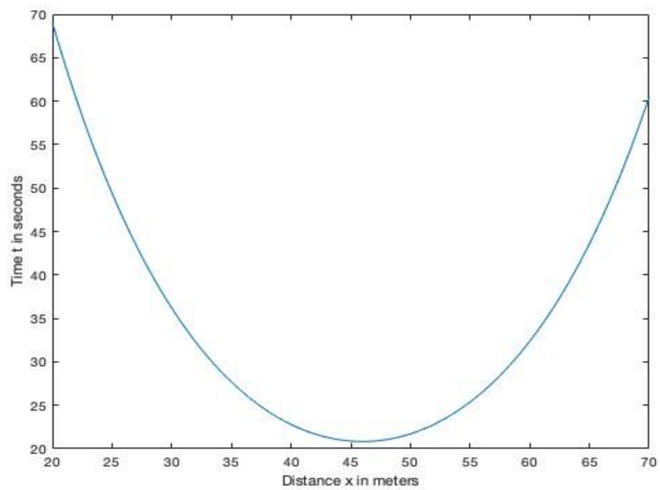

Figure 5. Results of $T(x_{\min})$ for Model 2 produced by MATLAB. Shows a minimum at x=45.9783 meters at t=20.8290 seconds. The parameters for this plot are: $x_1 - x_{01} = L - x$, $L = 90m$, $x_2 - x_{02} = x$, $v_{01} = 30.67 m/s$, $v_{02} = 35.28 m/s$, $\gamma = 0.01 kg/m$, $m = 0.14 kg$, and $\Delta t = 0.5s$.

3 Further Analysis

Here we look at a situation where we ignore air drag, so that, referring to Figure 3, the distance the ball travels due to player 1's throw is

$v_{01}t_1 = L - x$

and the distance the ball travels due to player 2's throw is

$v_{02}t_2 = x$,

where t_1 is the time the ball travels L-x and t_2 is the time of travel x. Thus, using $t_1 = (L-x)/v_{01}$ and $t_2 = x/v_{02}$, we have that the total time of projectile travel to home plate is

$$t_{\text{total}} = \frac{L}{v_{01}} + x\left(\frac{1}{v_{02}} - \frac{1}{v_{01}}\right) + \Delta t \,. \tag{25}$$

Thus, to check the results from the linear and quadratic models, we look at small drag coefficients. We use second order expansions; that is, $\ln(1+x) \sim x - x^2/2$ and $e^x \sim 1 + x + x^2/2$. Thus, in the limit of small c, equation (13b) gives

$$x(t) \approx x_0 + v_0 t - \frac{v_0 c t^2}{2m}. \tag{26}$$

so that as $c \to 0$, $x(t) \to x_0 + v_0 t$ and player 1's time $t_1 = \frac{x_1 - x_{01}}{v_{01}} = \frac{L - x}{v_{01}}$ while player 2's time is $t_2 = \frac{x}{v_{02}}$ for a total projectile time of $t_{\text{total}} = \frac{L}{v_{01}} + x\left(\frac{1}{v_{02}} - \frac{1}{v_{01}}\right) + \Delta t$, which is in agreement with the expected result of equation (25) in the limit of small c. In a similar fashion, for the quadratic model, using the expansion of the natural log this time, equation (20b) gives

$$x(t) \approx x_0 + v_0 t - \frac{\gamma v_0^2 t^2}{2m}. \qquad (27)$$

to identify that as $\gamma \to 0$, $x(t) \to x_0 + v_0 t$. We also see here that in the limit of zero γ, player 1's time is $t_1 = \frac{x_1 - x_{01}}{v_{01}} = \frac{L - x}{v_{01}}$ and player 2's time is $t_2 = \frac{x}{v_{02}}$ to obtain, once again the expected result of equation (25) when we add t_1, t_2, and Δt in the limit of small γ.

DISCUSSION

The purpose of this paper is to study the optimal ball launch performance by identifying optimal locations of players. Understanding the effects of air drag on a ball's projectile motion allows us to have a better grasp of the parameters that affect a ball's travel in sports. We investigated the full equations of motion for a ball in flight and studied two models (linear and quadratic) of decoupled equations of motion suitable for large speeds parallel to ground. We have applied the two models on a baseball scenario and the results from each of the models were tested using a hypothetical set of conditions. In the linear model, equation (19) yielded a minimum ball flight time of about 3.3 seconds at a distance x of about 77.2 meters. On the other hand, for the quadratic model, equation (24) gave a minimum ball flight time of about 20.8 seconds at a distance, x, of about 45.98 meters. The results from the linear and quadratic models, for the benefit of the reader, can be reproduced using MATLAB's scripts included in appendices A and B.

REFERENCES

Bogacki, P. and L. F. Shampine. 1989. A 3(2) pair of Runge-Kutta formulas, Applied Mathematics Letters, 2, 1-9

Goff, E. J. and M. Carre. 2009. Trajectory analysis of a soccer ball, American Journal of Physics, 77, 1020-1027.

Hasbun, J. E. 2008. Classical Mechanics wth MATLAB Applications, 1st Ed. Jones & Bartlett Learning. See also the 2nd. Ed. https://www.amazon.com/Classical-Mechanics-Applications-Javier-Hasbun/dp/1722299282m.

MATHEMATICA (https://www.wolfram.com/mathematica/), a technical computer software that is used, among other things, for solving differential equations.

MATLAB (https://www.mathworks.com), a matrix laboratory

ODE23 (https://www.mathworks.com/help/matlab/ref/ode23.html)

Magnus-Effect (https://www.britannica.com/science/Magnus-effect)

Mestre, N. D. 1990. The Mathematics of Projectiles in Sports, Cambridge, U. P

APPENDIX A

This is the script that was used to plot the hypothetical baseball scenario in Model 1. The values of the conditions are as follows: $L = 90\ m$; $V_{01} = 30.67\ m/s$; $V_{02} = 35.28\ m/s$; $c = 0.01\ kg/s$; $m = 0.14\ kg$; $\Delta t = 0.5\ s$.

----------Script Listing----------

```
%model_1.m Plot by Andrew Smith (July 2022) in collaboration with J. E.
%Hasbun
clear
m=0.14;             %Mass of baseball in kg
v01fs=100;          %speed of ball in ft/s
v02fs=115;          %speed of ball in ft/s
conv=3.26;            %number of ft in one meter
c=0.01;             %Given constant from original problem in kg/s
L=90;               %Total distance from player 1 to homeplate
Vo1=v01fs/conv;     %Velocity at which player 1 initially throws the ball
Vo2=v02fs/conv;     %Velocity at which player 2 initially throws the ball
delT=0.5;           %turn around time for player 2
x01=0;
xmin=x01+20;        %starting point
xmax=xmin+90.0;     %maximum distance
N=100; dx=(xmax-xmin)/N;
x=xmin:dx:xmax;     %distance array (independant var)
imax=length(x);     %Number of elements in distance array
for i=1:imax
    t(i)=-m*log((m*Vo1-c*(L-x(i)))/(m*Vo1))/c+delT-m*log((m*Vo2-c*x(i))/(m*Vo2))/c;
end
plot(x,t)

minx=(m*(Vo2-Vo1)/c+L)/2     %in meters
minxft=minx*conv             %in ft

mint=(-m/c)*(log((m*(Vo1+Vo2)-(c*L))/(2*m*Vo1))+log((m*(Vo1+Vo2)-
(c*L))/(2*m*Vo2)))+delT
```

APPENDIX B

This is the script that was used to plot the hypothetical baseball scenario in Model 2. The values of the conditions are as follows: $L = 90\ m$; $V_{01} = 30.67\ m/s$; $V_{02} = 35.28\ m/s$; $\gamma = 0.01\ kg/m$; $m = 0.14\ kg$; $\Delta t = 0.5\ s$.

----------Script Listing----------

```
%model_2.m by Andrew Smith (July 2022) in collaboration J. E.
%Hasbun
clear
m=0.14;               %mass of baseball
v01fs=100;            %speed of ball in ft/s
v02fs=115;            %speed of ball in ft/s
conv=3.26;            %number of ft in one meter
gam=0.01;             %Given constant from original problem
x01=0;                %player 1 position
%x02=                 %player 2 position
L=90;                 %Total distance from player 1 to homeplate
Vo1=v01fs/conv;       %Velocity at which player 1 initially throws the ball
Vo2=v02fs/conv;       %Velocity at which player 2 initially throws the ball
delT=0.5;             %turn around time for player 2
xmin=x01+20;          %starting point
xmax=xmin+50.0;       %maximum distance
N=100; dx=(xmax-xmin)/N;
x=xmin:dx:xmax;       %distance array (independant var)
imax=length(x);       %Number of elements in distance array
for i=1:imax
    t(i)=(m*((exp(gam*(L-x(i))/m)-1)/(Vo1*gam)))+delT+(m*((exp(gam*x(i)/m)-
1)/(Vo2*gam)));
end
plot(x,t)
minx=((m*log(Vo2/Vo1))/(2*gam))+L/2
minxft=minx*conv     %in ft
mint=m*((exp(gam*L/2/m)/sqrt(Vo1*Vo2)-1/Vo1)+exp(gam*L/2/m)/sqrt(Vo1*Vo2)-1/Vo2)/gam
+ delT
```